# UNIVERSITY COLLEGE LONDON

*University of London*

## EXAMINATION FOR INTERNAL STUDENTS

*For the following qualifications :-*

*B.A.* *B.Sc.* *M.Sci.*

**Mathematics B45: Discrete Mathematics for Computer Scientists**

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|---|---|---|
| COURSE CODE | : | **MATHB045** |
| UNIT VALUE | : | **0.50** |
| DATE | : | **08-MAY-02** |
| TIME | : | **14.30** |
| TIME ALLOWED | : | **2 hours** |

02-C0901-3-110



**TURN OVER**

*All questions may be attempted but only marks obtained on the best* **five** *solutions will count.*
*The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) Using a truth table, show that for any three sets $X$, $Y$ and $Z$,

$$(X \cup Y) \cap Z = (X \cap Z) \cup (Y \cap Z).$$

Draw a Venn diagram illustrating the set $(X \cup Y) \cap Z$.

(b) Prove that for any three sets $A$, $B$ and $C$,

$$(A \setminus B) \times C = (A \times C) \setminus (B \times C).$$

(c) Prove that $\mathbb{R}$ is uncountable.

2. (a) Define the terms *injection*, *surjection* and *bijection*.

Prove that every injective function has a left inverse.

(b) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as follows:

$$f(x) = \begin{cases} 2x+3 & \text{if } x > 0, \\ -x^2 & \text{if } x \leq 0. \end{cases}$$

Find a left inverse of $f$.

(c) Let $X$ be any set. Consider the function $g : X \times \mathbb{N} \to \mathbb{N}$, defined by $g(x, n) = n$.

For which sets $X$ is the function $g$ (i) injective, (ii) surjective, (iii) bijective.

3. (a) Consider the permutations

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 3 & 1 & 4 & 2 & 5 & 7 & 9 & 6 & 8 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 3 & 5 & 6 & 1 & 8 & 9 & 4 & 7 \end{pmatrix}.$$

Calculate $\sigma\tau$ and $\sigma^{-1}$.

Express $\sigma$ and $\tau$ in disjoint cycle notation.

Hence calculate $\sigma^{54}$ and $\tau^{878}$, expressing your answers in functional notation.

(b) Define the *dihedral group* $D_{2n}$.

List the elements of $D_8$, expressing each element first as a symmetry and then as a permutation in $S_4$.

Find the order of each element of $D_8$.

4. (a) Let $G$ be a finite group and let $H$ be a subgroup of $G$. Define a relation $\sim$ on $G$ by

$$a \sim b \text{ if and only if } ab^{-1} \in H.$$

Prove that the relation $\sim$ is an equivalence relation.

Let $[a]$ denote the equivalence class of $a$. Define a function $f : [a] \to H$ by $f(b) = ba^{-1}$. Prove that $f$ is bijective.

Hence deduce that $\#H$ divides $\#G$.

(b) Find an element of $S_5$ with order 6.

Hence write down a cyclic subgroup of $S_5$ with 6 elements.

5. (a) Solve the congruence

$$72x \equiv 2 \bmod 245.$$

(b) Define the *Euler Totient Function* $\varphi$.

Calculate $\varphi(245)$.

Hence calculate $72^{502} \bmod 245$.

(c) An R.S.A. public key is $(pq = 209, a = 103)$.
Using the factorization $209 = 11 \times 19$, find the private key $(pq, b)$.

Hence solve the congruence

$$y^{103} \equiv 2 \bmod 209.$$

6. (a) Find all real solutions to the following simultaneous equations by Gaussian elimination:

$$\begin{aligned} w + x + y + z &= 2, \\ w + 2x + 3y + 4z &= 5, \\ w + 3x + 5y + 8z &= 8. \end{aligned}$$

(b) Consider the matrix

$$A = \begin{pmatrix} 1 & 3 \\ 4 & 5 \end{pmatrix}.$$

Find a matrix $M$ such that $M^{-1}AM$ is diagonal.

Hence find a formula for $A^n$.

7. (a) A directed multigraph $\Gamma$ has the following incidence matrix:

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix}.$$

Draw the graph $\Gamma$.

Let $N(l)$ denote the number of circuits in $\Gamma$ of length $l$.

Find the recurrence relation satisfied by the numbers $N(l)$.

Calculate $N(l)$ for $l = 1, 2, 3, 4, 5$.

(b) Using the shortest path algorithm, find all the shortest paths between $A$ and $I$ in the following weighted graph.