UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C358: Cosmology

COURSE CODE : **MATHC358**

UNIT VALUE : **0.50**

DATE : **02-MAY-06**

TIME : **10.00**

TIME ALLOWED : **2 Hours**

06-C1018-3-50

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Evolution Equations:

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2 = \frac{H_0^2}{\rho_{c0}}\rho a^2. \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}a}(\rho a^3) = -3pa^2. \tag{2}$$

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)}; \qquad \rho_c \equiv \frac{3}{8\pi G}H^2; \qquad \Omega(t) \equiv \frac{\rho}{\rho_c}.$$

$$H_0 = h\,100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}.$$

Development angle/horizon coordinate:

$$\xi(t) \equiv \int_0^t \frac{\mathrm{d}t'}{a(t')}.$$

Robertson–Walker line element:

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - a^2(t)\left[\mathrm{d}\eta^2 + F^2(\eta)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right].$$

$$F(\eta) = \begin{cases} \sin\eta & k = +1 \\ \eta & k = 0 \\ \sinh\eta & k = -1 \end{cases}$$

1. Suppose the universe was radiation dominated for times t before the decoupling time a_d, with $p = \rho/3$.

   (a) Show that

   $$\rho(a) = \rho_d\left(\frac{a_d}{a}\right)^4$$

   where $a_d = a(t_d)$.

   (b) Solve the evolution equations for $k = 0$ to obtain $a(t)$.

   (c) Show that

   $$\rho(t) = \frac{3}{32\pi G}t^{-2}.$$

   (d) Suppose $\rho = N\alpha T^4$ where N is the effective number of radiative species and α is the Stefan-Boltzmann constant. Determine at what cosmic time the universe reaches the temperature T, i.e. find $t(T)$.

   (e) Suppose that nucleosynthesis ends when $T \approx 10^9\,\mathrm{K}$. Does the corresponding cosmic time $t(10^9)$ increase with N or decrease with N? Briefly describe how $t(10^9)$ affects the ratio of neutron density to proton density ρ_n/ρ_p? How does this ratio affect the present Helium abundance?

2. (a) Consider a galaxy emitting light at cosmic time t_1, with coordinates $(t_1, \eta_1, \theta_1, \phi_1)$. Suppose we observe this light at cosmic time t_0. Show that the ratio of the frequencies of observed to emitted light is

   $$\frac{\nu_0}{\nu_1} = \frac{a_1}{a_0}.$$

   (b) Express ν_0/ν_1 in terms of the redshift parameter z_1. Also express ν_0/ν_1 in terms of T_0/T_1, where T_0 is the present temperature of the microwave background, and T_1 is the temperature at time t_1.

   (c) For a $k = 0$ matter-dominated universe, the expansion parameter satisfies

   $$a(t) = a_0\left(\frac{3H_0 t}{2}\right)^{2/3}.$$

   Find $a(z)$, $t(z)$, and $r(z)$ for this universe.

   (d) The present record for furthest detected quasar is at $z = 6.4$. At approximately what cosmic time did the light observed from this quasar begin its journey (assuming h=2/3)?

3. (a) The Hubble parameter H_0 is often written $H_0 = h\,100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ (1 pc = 3.3 light years). Express $100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ in units of years^{-1}, showing your work. You need only be accurate to within 3% .

(b) Let the total mass-energy of the universe be $E(t) = \rho(t)\mathcal{V}(t)$. Show from the evolution equations that $E(t)$ satisfies the equation for adiabatic expansion,

$$dE = -p d\mathcal{V}.$$

(c) What is the 'entropy problem'? How could the inflationary universe model solve this problem? How could the anthropic principle solve this problem?

(d) What is meant by the terms *cold dark matter* and *hot dark matter*? If the early universe had been dominated by hot dark matter, why would we expect large clusters of galaxies? Give one example of a particle or object which could be the source of cold dark matter.

(e) Show that for $k = +1$,

$$\Omega(t) - 1 = \frac{1}{\dot{a}^2}.$$

Briefly, how does inflation in the very early universe bring Ω closer to the value 1?

4. Suppose the universe is static ($da/dt = 0$) and filled with a gas of mass density ρ, pressure p, velocity $\mathbf{v}$, and gravitational acceleration $\mathbf{F}$. The gas obeys the mass equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{v} = 0,$$

and the momentum equation

$$\rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \rho \mathbf{F}.$$

(a) Let $\rho = \bar{\rho} + \delta\rho$, $p = \bar{p} + \delta p$, $\mathbf{v} = \delta\mathbf{v}$ and suppose $\mathbf{F} = \delta\mathbf{F}$ where $\nabla \cdot \delta\mathbf{F} = -4\pi G \delta\rho$. Also let $c_s^2 \equiv dp/d\rho$. What is the physical meaning of c_s? Derive the equation

$$\frac{\partial^2 \delta\rho}{\partial t^2} - c_s^2 \nabla^2 \delta\rho - 4\pi G \bar{\rho} \delta\rho = 0.$$

(b) Show that this equation has solutions of the form

$$\delta\rho = A(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x} - \omega(k)t)}.$$

Determine $\omega(k)$, expressing your answer in terms of $k_J \equiv 4\pi G \bar{\rho}/c_s^2$. Let the wavelength of a density perturbation be $\lambda = 2\pi/k$. For what range of λ will a perturbation grow? What happens to the perturbation if λ is not in this range?

(c) Suppose a standing wave perturbation at some wavelength λ_1 starts at $t = 0$ with maximum amplitude, and reaches its next maximum amplitude at decoupling time $t = t_d$. Find an expression for λ_1 in terms of t_d, k_J and c_s.

5. (a) Consider light emitted at cosmic time t from a distant galaxy with cosmological redshift z. Show that the relation between t and z is given by

$$t(z) = \frac{1}{H_0} \int_z^{\infty} \frac{dz}{(1+z)E(z)},$$

where

$$E(z) = \frac{H(z)}{H_0}.$$

What is the present time t_0 in terms of this integral?

(b) In this problem we will assume that the universe is flat, i.e. $k = 0$. First consider a matter dominated universe, where $\rho = \rho_m$ and $\Omega_0 = \Omega_{m0}$. Show that

$$E(z) = E_m(z) = (1+z)^{3/2}.$$

(c) Using the integral expression for $t(z)$, calculate t_0 for a $k = 0$ matter dominated universe.

(d) Next consider a $k = 0$ universe with both matter and vacuum energy, so that $\Omega_0 = 1 = \Omega_{m0} + \Omega_{\Lambda 0}$. Derive an expression for $E(z)$ (call this function $E_{m\Lambda}(z)$).

(e) Show that the function $E_m(z)$ from part (b) is greater than the function $E_{m\Lambda}(z)$ from part (d) for $z > 0$ (and $\Omega_{\Lambda 0} > 0$). Does our estimate for the age of the universe increase or decrease if we observe a positive $\Omega_{\Lambda 0}$?