UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sc.* *M.Sci.*

Mathematics C358: Cosmology

COURSE CODE	:	**MATHC358**
UNIT VALUE	:	**0.50**
DATE	:	**10-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05-C0934-3-60

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TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Evolution Equations:

$$\dot{a}^2 + k = \frac{8\pi G}{3}\rho a^2 = \frac{H_0^2}{\rho_{c0}}\rho a^2. \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}a}(\rho a^3) = -3pa^2. \tag{2}$$

$$H(t) \equiv \frac{\dot{a}(t)}{a(t)}; \qquad \rho_c \equiv \frac{3}{8\pi G}H^2; \qquad \Omega(t) \equiv \frac{\rho}{\rho_c}.$$

$$H_0 = h\,100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}.$$

Development angle/horizon coordinate:

$$\xi(t) \equiv \int_0^t \frac{dt'}{a(t')}.$$

Robertson–Walker line element:

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - a^2(t)\left[\,\mathrm{d}\eta^2 + F^2(\eta)(\,\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\right].$$

$$F(\eta) = \begin{cases} \sin\eta & k = +1 \\ \eta & k = 0 \\ \sinh\eta & k = -1 \end{cases}$$

1. (a) Show that $\Omega > 1$ for a $k = +1$ universe, $\Omega = 1$ for a $k = 0$ universe, and $\Omega < 1$ for a $k = -1$ universe.

   (b) Show that $\rho(t) = \rho_0 a_0^3 / a^3(t)$ for a matter dominated universe.

   (c) The acceleration parameter Q_0 is defined by

$$Q_0 = \frac{\ddot{a}_0 a_0}{\dot{a}_0^2}.$$

   Suppose the universe is matter-dominated, so that $\Omega_0 = \Omega_{M0}$. Express Q_0 in terms of Ω_{M0}.

   (d) Now suppose both matter and vacuum energy are important. Show that the first evolution equation can be written

$$\dot{a}^2 + k = \frac{C}{a} + Da^2.$$

   What are the constants C and D? Rewrite this equation in the form

$$E = T + V,$$

   identifying which terms in the equation correspond to (total energy) E, (kinetic energy) T, and (potential energy) V. Sketch $V(a)$ on a graph, assuming that the maximum value of $V(a)$ is greater than -1. Give a brief explanation for why the $k = +1$ solution eventually contracts back to a big crunch.

2. Consider a galaxy of diameter D emitting light at coordinate r_1, redshift z_1, and time t_1 which we observe at time t_0.

   (a) Define the angular diameter distance d_A, and show that in terms of cosmic parameters,

$$d_A = a_0 r_1 (1 + z)^{-1}.$$

   (b) Let $E(z) = H(z)/H_0$. Find $E(z)$ for a $k = 0$ matter dominated universe.

   (c) The coordinate r_1 for a $k = 0$ universe can be calculated from the formula

$$r_1 = \frac{1}{H_0 a_0} \int_0^{z_1} \frac{1}{E(z)} \, dz$$

   (you do not need to show this). Find d_A as a function of redshift z.

   (d) The observed angular diameter δ of the galaxy depends on z_1. Assuming a fixed galactic size D, the angular diameter $\delta(z)$ has a minimum at some redshift z_{min}. What is z_{min} for a matter-dominated universe with $\Omega_0 = 1$?

   (e) In words, why should we expect that $\theta(z)$ has a minimum?

3. (a) What is a 'standard candle'? How does the Hertzsprung-Russell diagram provide us with standard candles? How do Cepheid variables provide us with standard candles?

   (b) What is the 'cosmic reference frame'? How does the *proper motion* (motion with respect to the cosmic reference frame) of galaxies, including the Milky Way, affect our observations? Why should we expect that galaxies have substantial proper motion?

   (c) What is decoupling time t_d? Why is this time significant for understanding the cosmic microwave background?

   (d) What is the 'horizon problem'? How does the inflationary universe model solve this problem? How does the cyclic universe model solve this problem?

   (e) Show from the evolution equations for a that

$$\frac{\mathrm{d}^2 a}{\mathrm{d}t^2} = \frac{-4\pi G}{3}(\rho + 3p)a. \tag{3}$$

   Use this equation, or otherwise, to conclude that a universe filled with ordinary matter and radiation cannot be static.

4. (a) Suppose $k \neq 0$. Show that at the present time t_0 the scale parameter a_0 satisfies

$$a_0 = \frac{1}{H_0\sqrt{|\Omega_0 - 1|}}.$$

   (b) For the rest of this question, suppose the universe is matter dominated with $k = +1$. The scale parameter is

$$a(\xi) = \frac{\Omega_0}{2H_0|\Omega_0 - 1|^{3/2}}(1 - \cos\xi)$$

   (you do not need to show this). Find ξ_0 as a function of Ω_0.

   (c) For what range of ξ is the universe collapsing?

   (d) Suppose that we are living at a time t_0 in the collapsing phase of the universe. Also suppose that $\Omega_0 = 2$. What is the value of ξ_0? Derive the redshift z as a function of ξ for $\xi < \xi_0$.

   (e) For what range of ξ is emitted light blueshifted?

5. Suppose a rotating black hole has angular momentum $J > 0$ and mass M. Let $a = J/M$ and $r_s = 2GM$. For motion in the equatorial plane ($\theta = \pi/2$), the Kerr metric line element in coordinates $(X^0, X^1, X^3) = (t, r, \phi)$ is

$$d\tau^2 = \left(1 - \frac{r_s}{r}\right) dt^2 + \frac{2ar_s}{r} dt\, d\phi - \frac{r^2}{r^2 - r_s r + a^2} dr^2 - \left(r^2 + a^2\left(1 + \frac{r_s}{r}\right)\right) d\phi^2.$$

(a) Consider a photon travelling in the radial direction ($d\phi = 0$). At which radii does $dt/dr = \infty$ along the photon path? What does the light cone look like at these radii?

(b) The area of a rotating black hole is

$$A = 4\pi(r_h^2 + a^2),$$

where the horizon radius $r_h = GM + \sqrt{G^2M^2 - a^2}$. Consider two identical rotating black holes, both of mass M and angular momentum J. Show that the sum of their areas is less than that of a single black hole of mass $2M$ and angular momentum $2J$. Can a single black hole split into two equal smaller black holes?

(c) Next consider a photon travelling in the ϕ or $-\phi$ direction ($dr = 0$). The *ergosphere* lies between the horizon and the radius r_{ergo} where $g_{00} = 0$. Show that a photon at $r = r_{erg}$ has only one possible non-zero value for $d\phi/dt$. What is this value? Show that if $a > 0$ the photon cannot travel in the $-\phi$ direction.

(d) Next consider a massive object moving on a geodesic in the equatorial plane. Show that there are conserved quantities $k = U_0$ and $h = -U_3$. Express these quantities in terms of $dt/d\tau$ and $d\phi/d\tau$. Show that an object with zero angular momentum ($h = 0$) must have positive angular velocity $d\phi/d\tau > 0$.