UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C358: Cosmology

COURSE CODE	:	**MATHC358**
UNIT VALUE	:	**0.50**
DATE	:	**07-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0923-3-60

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Evolution Equations:

$$\dot{R}^2 + k = \frac{8\pi G}{3}\rho R^2 = \frac{H_0^2}{\rho_{c0}}\rho R^2. \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}R}(\rho R^3) = -3pR^2. \tag{2}$$

$$H(t) \equiv \frac{\dot{R}(t)}{R(t)}; \qquad \rho_c \equiv \frac{3}{8\pi G}H^2; \qquad \Omega(t) \equiv \frac{\rho}{\rho_c}.$$

$$H_0 = h\,100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}.$$

Development angle/horizon coordinate:

$$\xi \equiv \int_0^t \frac{dt'}{R(t')}.$$

Robertson–Walker line element:

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - R^2(t)\left[\mathrm{d}\eta^2 + F^2(\eta)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)\right].$$

$$F(\eta) = \begin{cases} \sin\eta & k > 0; \\ \eta & k = 0; \\ \sinh\eta & k < 0. \end{cases}$$

1. (a) Let the mass-energy density and pressure of matter in the universe be ρ_m and p_m, with $p_m = 0$. Show that ρ_m satisfies

$$\rho_m(R) = \rho_{m0}\left(\frac{R}{R_0}\right)^{-3}.$$

Please give a brief explanation for why ρ_m is proportional to R^{-3}.

(b) Let the energy density and pressure of radiation in the universe be ρ_γ and p_γ, with $p_\gamma = \rho_\gamma/3$. Find $\rho_\gamma(R)$. Why does $\rho_\gamma(R)$ decrease more quickly with R than $\rho_m(R)$?

(c) Vacuum energy density ρ_Λ and pressure p_Λ satisfy $p_\Lambda = -\rho_\Lambda$. Find $\rho_\Lambda(R)$.

(d) Recent observations give $\Omega_{m0}/\Omega_{\gamma 0} \approx 3300$. At approximately what redshift was $\Omega_m(z) = \Omega_\gamma(z)$? Which type of energy density dominated the universe at higher redshifts?

(e) Consider the era when the universe was radiation dominated. Assume $\rho_m = \rho_\Lambda = 0$ and solve the evolution equations for $k = 0$ to find $R(t)$.

2. (a) Consider a galaxy emitting light at cosmic time t_1, with coordinates $(t_1, \eta_1, \theta_1, \phi_1)$. Suppose we observe this light at cosmic time t_0. Show that the ratio of the frequencies of observed to emitted light is

$$\frac{\nu_0}{\nu_1} = \frac{R_1}{R_0}.$$

(b) Express ν_0/ν_1 in terms of the redshift parameter z_1. Also express ν_0/ν_1 in terms of T_0/T_1, where T_0 is the present temperature of the microwave background, and T_1 is the temperature at time t_1.

(c) For a $k = 0$ matter-dominated universe, the expansion parameter satisfies

$$R(t) = R_0\left(\frac{3H_0 t}{2}\right)^{2/3}.$$

Find $R(z)$, $t(z)$, and $r(z)$ for this universe.

(d) The present record for furthest detected quasar is at $z = 6.4$. At approximately what cosmic time did the light observed from this quasar begin its journey (assuming h=2/3)?

3. (a) What is the Hawking area theorem for black holes? Show that a black hole of mass M and Schwarzschild radius $r_s = 2GM$ cannot split into two smaller black holes, each of mass $M/2$.

   (b) Briefly describe the inflation scenario. What is the flatness problem? How does inflation solve the flatness problem?

   (c) The Robertson–Walker metric line element given on the first page is expressed in terms of the coordinates (t, η, θ, ϕ). Convert this line element to coordinates (t, r, θ, ϕ) where $r = F(\eta)$. What is the geometrical significance of η? What is the geometrical significance of r?

   (d) The Schwarzschild metric line element is

$$\mathrm{d}\tau^2 = \left(1 - \frac{r_s}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{r_s}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2), \qquad r_s = 2GM.$$

   Consider two spacecraft at radii r_1 and r_2, with $r_1 > r_2 > r_s$. The two spacecraft have the same angular coordinates. The spacecraft at r_2 sends a message outwards at radio frequency ν_2, which is received at r_1 at frequency ν_1. Find the ratio ν_1/ν_2.
   Hint: For two wavefronts emitted at times t_2 and $t_2 + \Delta t_2$, consider the corresponding Δt_1, $\Delta\tau_1$, and $\Delta\tau_2$.

4. The flat T^3 three torus cosmology has metric line element

$$\mathrm{d}\tau^2 = \mathrm{d}t^2 - R^2(t)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2),$$

where

$$0 \le x < 1, \qquad 0 \le y < 1, \qquad 0 \le z < 1.$$

Here $R(t)$ is the periodicity length. Suppose this cosmology satisfies the matter-dominated $k = 0$ solution, $R_0 = 3 \times 10^9$ light years, and $h = 2/3$.

   (a) Briefly explain the terms *simply connected* and *multiply connected.*

   (b) In the three torus cosmology, light emitted by the sun long ago can circle all the way around the universe and come back to us. What is the redshift z_1 of sunlight which has circled the universe once in the x direction?

   (c) Discuss how observations might tell us that the universe is multiply connected.

5. Consider a fluid with mass density $\rho = \overline{\rho} + \delta\rho$, and velocity $\vec{v} = \delta\vec{v}$. Here $\overline{\rho} =$ constant, and fluctuating quantities are considered small. The continuity equation is

$$\partial_t \rho + \nabla \cdot \rho\vec{v} = 0.$$

We keep viscosity ν in the Navier–Stokes equation, so that

$$\partial_t \vec{v} + \vec{v} \cdot \nabla\vec{v} = -\frac{\nabla p}{\rho} + \delta\vec{F} + \nu\nabla^2\vec{v}.$$

The gravitational force satisfies

$$\nabla \cdot \delta\vec{F} = -4\pi G \delta\rho.$$

(a) Find a differential equation for $\delta\rho$.
Hint: terms will have up to three derivatives.

(b) Let

$$k_J^2 = \frac{4\pi G \overline{\rho}}{c_s^2},$$

where c_s is the sound speed. Assuming that solutions exist in the form

$$\delta\rho = A(\vec{k}) e^{i(\vec{k}\cdot\vec{x} - \omega t)},$$

show that

$$\omega^2 - c_s^2(k^2 - k_J^2) + i\nu k^2 \omega = 0.$$

(c) Solve this equation to find $\omega = \omega(k)$.

(d) Suppose we ignore viscosity, so that $\nu = 0$. What does $\omega(k)$ become? For what range of k do density fluctuations oscillate? For what range of k do they collapse?