UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualification:-

M.Sci.

Mathematics 3108: Modern Calculus of Variations

COURSE CODE	:	**MATH3108**
UNIT VALUE	:	**0.50**
DATE	:	**29-APR-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05-C0903-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

Throughout, we will assume that $F \in C^2(\mathbb{R}^3)$, $[a,b] \subset \mathbb{R}$ and $A, B \in \mathbb{R}$ are given. For absolutely continuous functions u on $[a,b]$ denote

$$\mathcal{F}(u) = \int_a^b F(x, u(x), u'(x))\, dx$$

1. (a) State the Euler–Lagrange equation and show that it holds for any minimizer of the problem

$$\textit{minimize } \mathcal{F}(u) \textit{ over } u \in C^2[a,b], u(a) = A, u(b) = B.$$

   (b) Describe how the method of calibrators may be used to show that a solution of the Euler–Lagrange equation is a minimizer. (You may be assume that F is convex in the last variable.)

2. (a) Define the notion of absolute continuity for functions $u : [a,b] \to \mathbb{R}$. Also explain what is meant by saying that a function from $L^1[a,b]$ is the distributional derivative of u.

   (b) Show that u is absolutely continuous if and only if it has distributional derivative belonging to $L^1[a,b]$.

3. (a) Suppose that $F(x,y,p)$ is convex in p and has super-linear growth. Show that the problem

$$\textit{minimize } \mathcal{F}(u) \textit{ over } u \in \mathrm{AC}[a,b] \textit{ such that } u(a) = A,\ u(b) = B$$

   has a solution.

   (b) Show that this conclusion may fail if we assume only that $F(x,y,p)$ is convex in p and that $F(x,y,p) \geq |p|$.

4. (a) State Tonelli's partial regularity theorem.

   (b) Prove its statement concerning existence of the first derivative provided that you already know that under the assumptions of the theorem, every minimizer of the problem has the following property:

   Whenever $a \le x_i \le t_i < s_i \le y_i \le b$ are such that

$$y_i - x_i \to 0 \quad \text{and} \quad \frac{u(y_i) - u(x_i)}{y_i - x_i} \to c \in \mathbb{R},$$

   then

$$\frac{u(s_i) - u(t_i)}{s_i - t_i} \to c.$$

5. Show that for $F(x, y, p) = K(x^3 - y^5)^2 p^{20} + p^2$ with a suitable $K > 0$, every solution of the problem

$$\textit{minimize } \mathcal{F}(u) \textit{ over } u \in \mathrm{AC}[-1, 1] \textit{ such that } u(-1) = -1,\ u(1) = 1$$

satisfies $u'(0) = \infty$.