UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C326: Boundary Layers

COURSE CODE : **MATHC326**

UNIT VALUE : **0.50**

DATE : **12-MAY-06**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

06-C1007-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

For questions 2 to 5, assume two-dimensional incompressible laminar flow.

The standard dimensional boundary layer equations for a boundary layer in the neighbourhood of $y=0$ are

$$u_x + v_y = 0 \quad , \quad u_t + uu_x + vu_y = U_t + UU_x + \nu u_{yy} \quad .$$

Here u and v are velocity components in the x and y directions respectively, t is time, ν is the kinematic viscosity of the fluid, and $U(x,t)$ is the external flow in the x direction at $y=0$. Subscripts denote partial derivatives.

The streamfunction ψ is defined such that $u=\psi_y$ and $v=-\psi_x$.
For length scale L and velocity scale U_0, the Reynolds number is $R=U_0L/\nu$.

1. A two point boundary value problem for the function $h(x)$ is defined by the differential equation

$$\epsilon(1+3x)h_{xx} - h = -C \quad ,$$

where ϵ is a small positive parameter, C is a constant, and the boundary conditions are

$$h(0) = 1 \ , \quad h(1) = 2 \ .$$

What property makes this an example of a singular perturbation problem?

(a) For the case $C=2$, determine the location of a boundary layer and find the first terms of the outer and inner asymptotic expansions.
Provide a sketch of the resulting leading order solution for $h(x)$.
Find the second term of the inner expansion.

(b) For the case $C=1$, again find the location of the boundary layer and the first terms of the inner and outer expansions.

2. What is a basic property of flow represented by a similarity solution?

(a) Consider the standard boundary layer equations for steady flow in the case $U(x) = Cx^m$, where $C > 0$ and m are constants. By choosing a similarity variable $\eta = y/(Ax^n)$, where A and n are other constants, and a streamfunction of the form

$$\psi(x, y) = ACx^{m+n}f(\eta) ,$$

show that a similarity solution can be found if $2n = 1 - m$. (Note that m and n need not be integers.)

By choosing $A^2 = \nu/[C(m+n)]$, derive the Falkner-Skan equation

$$f''' + ff'' + m(1 - f'^2)/(m+n) = 0 .$$

When $C = U_0/L^m$ (i.e. $U(x) = U_0(x/L)^m$), show that the similarity variable can be written as

$$\eta = R^{1/2}(m+n)^{1/2}(y/L)(x/L)^{-n} .$$

(b) The streamfunction for a steady inviscid flow in polar co-ordinates is

$$\psi = (U_oL/\lambda)(r/L)^\lambda \; sin[\lambda(\theta - \pi)] ,$$

where λ=3/2, and the constants U_0 and L are velocity and length scales respectively.

Given that $u^{(r)}=\psi_\theta/r$ and $u^{(\theta)}=-\psi_r$, find expressions for these velocity components.

Show that the lines $\theta=\pi/3$, $\theta=\pi$ and $\theta=5\pi/3$ are streamlines.

Sketch the streamlines in the region $\pi/3 \leq \theta \leq 5\pi/3$.

(c) How might the result in (a) be used to analyse the boundary layers for flow past a wedge, for which the inviscid flow is that given in (b)? What would be appropriate values for m and n in this case?

3. Consider steady flow past a flat plate that lies along $y=0$, between $x=-L$ and $x=0$. The flow is symmetric about $y=0$, and upstream of the plate the flow is $(u,v)=(U_0,0)$ where U_0 is a positive constant.

(a) In the wake downstream of the plate the standard boundary layer equations can be used. From those equations, prove that the quantity

$$\int_0^\infty u(U_0 - u)\,dy$$

is independent of x.

(b) Far downstream, suppose $u/U_0 = 1 - F(x,y)$, where $F \ll 1$. Derive the linearised boundary layer equation

$$F_x = A^2 F_{yy} \; ,$$

where $A^2=\nu/U_0$.

(c) By using the substitution $F=f(\eta)/x^{1/2}$, where $\eta=y/(Ax^{1/2})$, derive the ordinary differential equation

$$f'' + (\eta f' + f)/2 = 0 \; .$$

Justify the boundary conditions $f'=0$ at $\eta=0$ and $f \to 0$ as $\eta \to \infty$.
Find f to within an arbitrary multiplicative constant.
How might the value of that constant be determined?

4. Suppose flow past an obstacle is impulsively started from rest at time $t=0$, such that the inviscid flow at the surface of the obstacle, along $y=0$, is given to be $U(x)$ for $t > 0$.

Given that the streamfunction for the flow in the boundary layer has the form

$$\psi = 2(\nu t)^{1/2} \left[UF_0(\eta) + tUU_xF_1(\eta) + \text{terms of order } t^2 \right] ,$$

where $\eta = y/(4\nu t)^{1/2}$, find corresponding expressions for u and v.

(a) Derive the ordinary differential equation

$$F_0''' + 2\eta F_0'' = 0$$

for F_0, and also derive three boundary conditions.

(b) Given that

$$\int_0^\infty e^{-\lambda^2} d\lambda = \sqrt{\pi}/2 ,$$

prove that to leading order

$$u = U(x)(2/\sqrt{\pi}) \int_0^\eta e^{-\lambda^2} d\lambda .$$

(c) For the case $U(x) = U_0(1 + e^{-x^2/L^2})$, find the value of x where separation is first expected to occur on the surface of the obstacle. Hence show that separation is first expected at time

$$t = (L/U_0)\,(2e/\pi)^{1/2} \,/\, F_1''(0) .$$

(You may assume that $F_1''(0)$ is positive.)

5. With velocity scaled by U_0, distance scaled by L, and pressure scaled by $\rho U_0{}^2$, dimensionless equations for steady flow are

$$uu_x + vu_y = -p_x + R^{-1}(u_{xx} + u_{yy}) \quad , \qquad (1)$$

$$uv_x + vv_y = -p_y + R^{-1}(v_{xx} + v_{yy}) \quad , \qquad (2)$$

with $R^{-1} \ll 1$, and $u_x + v_y = 0$. Consider flow past a flat plate lying along y=0, between x=−1 and x=0, with (u, v)=(1, 0) far from the plate.

In the context of matched asymptotic expansions, why is a boundary layer required near the plate?

If the inner variable is Y=y/ϵ, state (without proof, but giving a reason) the choice of the small parameter ϵ required to obtain the standard boundary layer equations.

(a) For the triple deck theory required near the trailing edge of the plate, appropriate scales for the upper deck are W=y/δ^3 and X=x/δ^3, with expansions of the form

$$\psi \sim y - \delta^4 C + \delta^5 F(X, W) \quad ,$$
$$p \sim \delta^2 P(X, W) \quad ,$$

where δ^4=R^{-1} and C is a constant.
What does the term $-\delta^4 C$ represent?
With these scalings, rewrite (1) and (2) in terms of F and P.
Deduce that F_{WX}=$-P_X$ and F_{XX}=P_W .
Given $V(X, W) = -F_X = f(X)g(W)$, show that

$$V_{XX} + V_{WW} = f_{XX}g + fg_{WW} = 0 \quad .$$

(b) The boundary conditions for V are $V \to 0$ as $X \to \pm\infty$ and $W \to \infty$, and $V \to -A_X$ as $W \to 0$ for some function $A(X)$. Prove that

$$V = -(2\pi)^{-1} \int_{-\infty}^{\infty} A_\lambda \frac{2W}{W^2 + (X - \lambda)^2} d\lambda \quad .$$

You are given that

$$\int_{-\infty}^{\infty} e^{ik(X-\lambda)} e^{-|k|W} dk = \frac{2W}{W^2 + (X - \lambda)^2} \quad ,$$

and you may use the Fourier transform relations

$$\hat{f}(k) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} f(X) e^{-ikX} dX \quad , \qquad f(X) = (2\pi)^{-1/2} \int_{-\infty}^{\infty} \hat{f}(k) e^{ikX} dk \quad .$$

(c) Given that $P \to 0$ as $W \to \infty$, and that V_X=$-P_W$, use the above expression for V to find a similar expression for P.

END OF PAPER