**UNIVERSITY COLLEGE LONDON**

University of London

**EXAMINATION FOR INTERNAL STUDENTS**

For The Following Qualification:-

*M.Sci.*

**Mathematics C326: Boundary Layers**

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| COURSE CODE | : | **MATHC326** |
| UNIT VALUE | : | **0.50** |
| DATE | : | **24-MAY-04** |
| TIME | : | **14.30** |
| TIME ALLOWED | : | **2 Hours** |

04-C0918-3-30


**TURN OVER**

*All questions may be attempted but only marks obtained on the best* **four** *solutions will count.*
*The use of an electronic calculator is* **not** *permitted in this examination.*

For questions 2 to 5, assume two-dimensional incompressible laminar flow.

The dimensional boundary layer equations for a boundary layer in the neighbourhood of $y = 0$ are

$$u_x + v_y = 0 \quad , \quad u_t + uu_x + vu_y = U_t + UU_x + \nu u_{yy} \quad .$$

Here $u$ and $v$ are velocity components in the $x$ and $y$ directions respectively, $t$ is time, $\nu$ is the kinematic viscosity of the fluid, and $U(x,t)$ is the external flow in the $x$ direction. Subscripts denote partial derivatives.

The streamfunction $\psi$ is defined such that $u = \psi_y$ and $v = -\psi_x$.

1. Consider the differential equation

$$\epsilon\, d^2h/dx^2 - x\, dh/dx + Nh = 0$$

with boundary conditions $h = 1$ at $x = 1$, $h = 2$ at $x = 2$. Here $\epsilon$ is a small positive parameter, and $N$ is a positive integer. A solution is required in the domain $1 \le x \le 2$.

(a) Where is a boundary layer likely to occur, and why?

(b) Find a suitable inner variable, and find the leading term in the outer and inner expansions. Match the expansions to determine any unknown coefficients. Sketch the inner and outer solutions to this order for $N$=2.

(c) Find a composite expansion to this order. Comment on the solution when $N$=1.

(d) For general $N$, find the differential equation and relevant boundary condition for the next term in each of the inner and outer expansions.

2. With reference to the Navier-Stokes equations, explain (without proof) in what way the boundary layer equations can be regarded as an 'inner' expansion matching to an inviscid 'outer' expansion.

   The radial velocity component in an inviscid steady sink flow field is given by $u^{(r)}(r,\theta) = -U_oL/r$ , where $U_o$ and $L$ are (positive) velocity and length scales respectively, and $r$ and $\theta$ are usual polar co-ordinates. In Cartesian co-ordinates the sink is at $x = y = 0$. Suppose there is a flat plate at $y = 0$, and we wish to investigate the boundary layer induced by viscous effects for the region $x > 0$ and $x \sim 1$.

   (a) Show that the relevant external flow for the boundary layer equations is $U(x) = -A/x$, where $A=U_oL$.

   (b) Show that the boundary layer equations have a solution of the form
   $$\psi = -(\nu A)^{1/2} f(\eta) \quad \text{with} \quad \eta = (y/x)(A/\nu)^{1/2} \quad ,$$
   where
   $$f''' = f'^2 - 1 \quad .$$

   (c) State two suitable boundary conditions for $f'$.

   (d) Find the shear stress on the plate as a function of $x$. (Note: it can be shown that $f''^2 = (2/3)(f'-1)^2(f'+2)$ .)

3. Suppose a symmetric smooth blunt body is located in an external flow field with the same symmetry. Explain what is meant by the term 'front stagnation point'.

   With the front stagnation point at $x = 0$ in a steady flow, suppose
   $$U(x) = U_o\,[\,(x/L) - (x/L)^3\,]$$
   where $U_o$ and $L$ are constants.

   Try a solution to the boundary layer equations of the form
   $$\psi = \epsilon U_oL\,[\,(x/L)F_1(\eta) - 4(x/L)^3F_3(\eta) + \text{further terms}] \quad , \quad \eta = y/(L\epsilon)$$
   where $\epsilon^2 = \nu/U_oL$. (You may find it convenient to define and use a new variable $\hat{x} = x/L$.)

   (a) Show that $F_1$ satisfies the nonlinear ordinary differential equation
   $$F_1''' + F_1''F_1 - F_1'^2 + 1 = 0 \quad .$$

   (b) Find another ordinary differential equation involving $F_3$ and $F_1$, by equating $(x/L)^3$ terms .

   (c) What are appropriate boundary conditions for $F_1$ and $F_3$?

   (d) In terms of the second derivatives of $F_1$ and $F_3$ at $\eta = 0$ (which are both positive), estimate (to this level of approximation) the value of $x/L$ where the surface shear stress is zero.

4. Suppose a bluff body is in a flow that accelerates uniformly from rest at $t = 0$. Along the surface of the body the inviscid flow is

$$U(x,t) \;=\; t\,A(x) \quad \text{for} \;\; t > 0 \;\; .$$

This situation is similar to that of impulsively started flow, and approximate solutions of the bounday layer equations can be obtained using expansions in powers of $t$. Suppose the first two terms in the expansion for $\psi$ are

$$\psi \;=\; 2(\nu t)^{1/2}\,[\; tAF_1(\eta) \;+\; t^3 AA_x F_3(\eta) \;]$$

where

$$\eta \;=\; y \,/\, [2(\nu t)^{1/2}]$$

(a) By differentiating $\psi$, find the corresponding expressions for $u$ and $v$.

(b) Show that the ordinary differential equation (ODE) for $F_1$ is

$$F_1''' \;+\; 2\eta F_1'' \;-\; 4F_1' \;+\; 4 \;=\; 0 \;\; .$$

(c) Which terms in the boundary layer equations balance to this order? What are the appropriate boundary conditions for the ODE?

(d) When and where is separation first likely to occur? (You may assume that $F_1''(0)$ and $F_3''(0)$ are positive.)

5. For this question, use the nondimensional steady continuity and Navier-Stokes equations in the form

$$\begin{aligned} u_x + v_y &= 0 && (1), \\ uu_x + vu_y &= -p_x + R^{-1}(u_{xx} + u_{yy}) && (2), \\ uv_x + vv_y &= -p_y + R^{-1}(v_{xx} + v_{yy}) && (3). \end{aligned}$$

where $R$ is the Reynolds number.

Consider uniform flow past a flat plate located at $y = 0$ in the region $-1 \leqslant x \leqslant 0$. Upstream of the plate the flow has $u = 1$ , $v = 0$. Why are the boundary layer equations not valid near the trailing edge at $x = 0$?

Let $\delta = R^{-1/8}$. Use the triple-deck scalings $X = x/\delta^3$ , $Z = y/\delta^5$ (lower deck), $Y = y/\delta^4$ (middle deck), $W = y/\delta^3$ (upper deck).

Assume the following expansions:

$$\begin{aligned} &\text{upper deck} \quad u \sim 1 + \delta^2\bar{u}(X, W) \ , \ p \sim \delta^2\bar{P}(X, W) \ , \\ &\text{middle deck} \quad u \sim u_o(Y) + \delta u_1(X, Y) \ , \ p \sim \delta^2 P(X, Y) \ , \\ &\text{lower deck} \quad u \sim \delta u_L(X, Z) \ , \ p \sim \delta^2 P_L(X, Z) \end{aligned}$$

(a) In each deck, find the leading balance of terms in the Navier-Stokes equation (2) above. In the upper deck, find also the leading balance in equation (3). (Hint: in each deck first use the continuity equation to determine the order of $v$.)

(b) By considering equation (3), deduce that $P = P(X)$ in the middle deck, and $P_L = P_L(X)$ in the lower deck.

(c) What are the boundary conditions for velocity at $Z$=0?

(d) What are the boundary conditions on $u$ as $X \to -\infty$ in each deck? (Assume the flow leading up to the vicinity of the trailing edge is given by the known Blasius flow denoted $u_B(x, y)$.)