UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

M.Sci.

Mathematics C326: Boundary Layers

COURSE CODE	:	**MATHC326**
UNIT VALUE	:	**0.50**
DATE	:	**29-APR-02**
TIME	:	**10.00**
TIME ALLOWED	:	**2 hours**

02-C0909-3-30

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

In questions 2, 3, 4 the required boundary-layer equations for a boundary layer in the neighbourhood of $y = 0$ are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{\partial U}{\partial t} + U\frac{\partial U}{\partial x} + \nu\frac{\partial^2 u}{\partial y^2}.$$

Here u, v are the velocity components in the x, y directions, t is the time, ν is the kinematic viscosity of the fluid and $U(x,t)$ is the external flow in the x-direction. The streamfunction ψ is defined by $u = \partial\psi/\partial y$, $v = -\partial\psi/\partial x$.

1. Explain the meaning of the terms 'regular' and 'singular' as applied to a perturbation solution of a differential equation.
The equation

$$\varepsilon y'' + (1 + e^{-x})yy' + e^{-x}y = 0$$

has boundary conditions $y(0) = 0$, $y(\infty) = 1$, and ε is a small positive parameter. Find the leading term of $y'(0)$ and give a sketch of the form of your solution for all $x \geq 0$.

2. Explain the principle steps in the argument which leads to the steady form of the boundary-layer equations given above with boundary conditions

$$u = v = 0 \text{ on } y = 0, \qquad u(x, \infty) = U(x).$$

The mainstream for a sink-like boundary-layer flow in the neighbourhood of a wall $y = 0$ is $U(x) = -k/x$ where k is a positive constant. Show that the steady boundary-layer equations have a solution in which

$$u = \frac{-k}{x}f'(\eta) , \qquad \eta = x^q y/(c\nu)^{\frac{1}{2}}$$

with

$$f''' + 1 - f'^2 = 0 , \quad f(0) = f'(0) = 0 , \; f'(\infty) = 1,$$

if the constants q and c in the similarity variable η are correctly chosen. Multiply through by $f''(\eta)$ and integrate this equation for $f'(\eta)$ **once** to find the skin friction, and give an expression for the drag on the interval $L \leq x < \infty$ of the plate.

3. Sketch the streamlines $\psi =$ constant when $\psi = axy$, a being a positive constant. Show that these are the inviscid streamlines for a stagnation point at $x = y = 0$, and that if there is a boundary at $y = 0$, this inviscid flow gives a slip velocity $U(x) = ax$.
The solution in the boundary layer on $y = 0$ is required. Show that the steady boundary-layer equations have a solution in which

$$u = axf'(\eta) , \qquad v = -(a\nu)^{\frac{1}{2}} f(\eta) , \qquad \eta = y\left(\frac{a}{\nu}\right)^{\frac{1}{2}}$$

and give the similarity equation satisfied by $f(\eta)$.
State the boundary conditions on f if the wall is impermeable and the no-slip condition is to be satisfied.
Suppose now that the wall is permeable and suction is applied so that $f(0) = f_w >> 1$, but the no-slip condition is still to be satisfied. Make the change of variables

$$f(\eta) = f_w + (f_w)^{-1}\varphi(\xi) , \qquad \xi = \eta f_w$$

and write down the limiting form of the similarity equation and boundary conditions as $f_w \to \infty$. Find the solution for $\varphi'(\xi)$ and hence show that

$$u(x,y) = ax(1 - e^{v_w y/\nu})$$

where $v_w(< 0)$ is the value of v on the wall.

4. Show that a uniform stream U_∞ of inviscid fluid past a fixed circular cylinder of radius a gives a slip velocity $U(x) = 2U_\infty \sin(x/a)$, where x measures distance around the cylinder from the front stagnation point.
It is assumed that at time $t = 0$ a boundary-layer flow with mainstream $U(x)$ is impulsively set up around this cylinder. Show that, for small time, the two-dimensional boundary-layer equations admit a solution for the stream function of the form

$$\psi(x,y,t) = 2(\nu t)^{\frac{1}{2}} \left\{ U\zeta_0(\eta) + tU\frac{dU}{dx}\zeta_1(\eta) + O(t^2) \right\} ,$$

where $\eta = \dfrac{y}{2(\nu t)^{\frac{1}{2}}}$. Here y measures distance along the normal to the cylinder.
Show that

$$\zeta_0'(\eta) = \frac{2}{\sqrt{\pi}} \int_0^\eta e^{-\lambda^2} d\lambda$$

and that $\zeta_1(\eta)$ satisfies

$$\zeta_1''' + 2\eta\zeta_1'' - 4\zeta_1' = 4(\zeta_0'^2 - \zeta_0\zeta_0'' - 1), \qquad \zeta_1(0) = \zeta_1'(0) = \zeta_1'(\infty) = 0.$$

Given that $\zeta_1''(0) = \dfrac{2}{\sqrt{\pi}}\left(1 + \dfrac{4}{3\pi}\right)$, show that separation first occurs at the rear stagnation point and find the approximate time at which it occurs.

5. The interactive boundary-layer equations

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{dp}{dx} + \frac{\partial^2 u}{\partial y^2}\ , \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

are to be solved with the boundary conditions

$$u = v = 0 \quad \text{on} \quad y = 0\ ,$$

$$u - y \to -\, p(x) \quad \text{as} \quad y \to \infty.$$

Write $u = y + ae^{kx}f'(y)$, $v = -ake^{kx}f(y)$, $p = ap_0e^{kx}$ where $a, p_0, k(k > 0)$ are constants, and show that, if $|a| << 1$, so that a^2 may be neglected, the equations are satisfied if

$$f''' + kf - kyf' = p_0k$$

and that the boundary conditions for f are

$$f(0) = f'(0) = 0\ ; \quad f'(\infty) = -p_0.$$

Show that $f'''(0) = p_0k$, and differentiate the above equation to yield a second order equation for $f''(y)$. Given that the solution of the equation $d^2F/dY^2 - YF = 0$ which is exponentially small at $Y = \infty$ is an Airy function $\mathrm{Ai}(Y)$, show that $f''(y) = C\mathrm{Ai}(k^{\frac{1}{3}}y)$ where C is a constant. Use the boundary conditions to determine the constant k.

[You may take $\mathrm{Ai}'(0) = -\frac{1}{4}$ and $\int_0^\infty \mathrm{Ai}(x_1)dx_1 = \frac{1}{3}$.]

END OF PAPER