UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics C394: Biomechanics

COURSE CODE	:	**MATHC394**
UNIT VALUE	:	**0.50**
DATE	:	**23-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0926-3-50

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. In each of the following state clearly any additional scaling assumptions you make.

   (a) A human delivers a punch with his arm. Show that the speed at which his fist is moving just before it hits its object is scale independent.

   (b) Give a simple scaling argument to show that the heart rate of an animal should scale like ℓ^{-1}, where ℓ is a linear dimension.

   (c) A long jumper jumps from a standing start. Show that the maximum horizontal distance she can jump is scale independent. You may assume that air resistance is negligible.

   (d) An animal jumps from the top of a vertical wall of height h. Show that the height at which the animal can just escape breaking its legs on landing is scale independent. You may assume that air resistance is negligible during the jump.

2. The probability that a given bone of an individual animal in a population experiences a bending moment in the infintesimal interval $[C, C+\delta C)$ during its lifetime is $Q(C)\delta C$ for some density function $Q(C)$. If $\bar{C}$ is the expected bending moment experienced, and the bone can withstand a maximum bending moment C_{max} before breaking, then the bone is said to have *safety factor* $s = C_{max}/\bar{C}$.

The total *cost* to an animal of maintaing such a bone splits into two components, $\mathcal{C}(s) = \mathcal{C}_{grow}(s) + \mathcal{C}_{fail}(s)$, where

$$\mathcal{C}_{grow} = cs^{\frac{2}{3}}, \qquad \mathcal{C}_{fail} = cVP(s),$$

with c and V positive constants, and $P(s)$ the probability that the animal will break its bone during its lifetime.

(a) Explain why $\mathcal{C}_{grow}$ is proportional to $s^{\frac{2}{3}}$, and how $P(s)$ may be obtained from $Q(C)$. Also explain why V can be regarded as a measure of the 'value' of the bone to the animal.

[If the bone has outer radius r and inner radius radius kr, with $0 < k < 1$, you may assume the formula

$$C_{max} = \frac{\pi b}{4}(1-k^4)r^3.]$$

(b) If $Q(C) = \alpha^2 C e^{-\alpha C}$ for some $\alpha > 0$, show that

$$P(s) = (1+2s)e^{-2s}.$$

Deduce that $P(s)$ is monotonically increasing for $V < \hat{V} = \frac{1}{6}\left(\frac{3e}{2}\right)^{\frac{4}{3}}$, and that, for $V > \hat{V}$, there is a unique $s^* > \frac{2}{3}$, at which $\mathcal{C}(s)$ has a local minimum.

3. (a) A human runner of mass M runs on flat horizontal ground at constant horizontal speed u, and with constant step length s. Obtain the expression

$$\mathrm{WD} = \frac{Mg^2s^2}{8u^2}$$

for the work done by the runner's leg muscles in launching each step. Explain the simplifying assumptions you use.

(b) The runner encounters a flat hill at an angle α to the horizontal. The runner maintains her step length s, and the speed u, parallel to the hill, at which her foot leaves the ground in each step. Under the same simplifying assumptions as used in (a), show the following.

(i) The time T between successive footfalls is

$$T = \frac{u}{g}\mathrm{cosec}\alpha \left\{ 1 - \sqrt{1 - \frac{2sg}{u^2}\sin\alpha} \right\}.$$

(ii) The runner's speed parallel to the hill just *before* her foot hits the ground is

$$u_- = u\sqrt{1 - \frac{2sg}{u^2}\sin\alpha}.$$

(iii) The runner's speed perpendicular to the hill just *after* her foot leaves the ground is

$$v = \frac{gT}{2}\cos\alpha.$$

(iv) The work done by the runner's leg muscles in launching each step is

$$\mathrm{WD} = \frac{1}{2}Mv^2 + Mgs\sin\alpha.$$

4. (a) If a bird flies at constant speed u, the total drag force it experiences is given by the formula

$$D = \frac{1}{2}\rho C_{Df} S u^2 + \frac{KL^2}{\frac{1}{2}\rho A_w u^2}.$$

Briefly explain the terms in this formula.

(b) During the downward stroke of a bird's wings in powered flight, the wing acquires a vertical velocity v. Show that v is scale independent. Also show that the bird is capable of producing a maximum power output which scales like ℓ^2, where ℓ is a linear dimension.

(c) A bird maintains powered flight at constant speed u in an upward direction at constant angle θ to the horizontal. Draw a diagram illustrating the forces acting on the bird, and obtain an expression for the power required to maintain this flight.

Show that there is a speed u^* at which the power required, P_R, is a minimum, and that u^* scales like $\ell^{\frac{1}{2}}$ and $P_{R,min}$ scales like $\ell^{3.5}$. Explain why this result means that very large birds cannot maintain this kind of flight for any angle $\theta \geqslant 0$.

5. (a) A bacterium moves in a 3-dimensional medium. The probability that the bacterium is in a small volume δV containing the point with position vector $\mathbf{r}$ at time t, is $p(\mathbf{r}, t)\delta V$. The probability density $p(\mathbf{r}, t)$ satisfies the diffusion equation

$$\frac{\partial p}{\partial t} = -\nabla \cdot [\mathbf{k}(\mathbf{r})p] + \frac{1}{2}\Delta [m(\mathbf{r})p],$$

where Δ is the 3-dimensional Laplace operator. Briefly explain the assumptions underlying this equation, and the meanings of the functions $\mathbf{k}(\mathbf{r})$ and $m(\mathbf{r})$.

(b) A bacterium lives in a pond containing homogeneous water. The pond is infinitely deep, and occupies the region $z \leqslant 0$, with its surface at $z = 0$. Light reaches the surface from $z = \infty$, and has uniform surface intensity ϕ_0. The light intensity in the water declines with depth according to the function

$$\phi(z) = \frac{\phi_0}{1 - z} \qquad (z \leqslant 0).$$

The bacterium is sensitive to light and preferentially moves towards regions of high light intensity. The probability density function for the depth at which the bacterium can be found, $p(z, t)$, satisfies

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial z}[\phi(z)p] + \frac{1}{2}m\frac{\partial^2 p}{\partial z^2}.$$

Show that there is a steady-state solution with $p(z) \to 0$ and $p'(z) \to 0$ as $z \to -\infty$, only if $\phi_0 > \frac{1}{2}m$, and find this solution explicitly.