UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M13B: Applied Mathematics 2

COURSE CODE	:	**MATHM13B**
UNIT VALUE	:	**0.50**
DATE	:	**03-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1040-3-110

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. A particle of soot of mass m moves vertically upwards against gravity and a retarding force $mg\lambda v^{2/N}$ where v is the particle speed and λ, N are positive constants.

   (a) Show that the particle will travel a distance

   $$g^{-1}\int_0^{v_0}\frac{vdv}{(\lambda v^{2/N}+1)}$$

   before coming momentarily to rest, if its initial speed is v_0.

   (b) In the case $N=3$, deduce by means of a substitution or otherwise that the distance travelled is

   $$3\left\{3+2\ln(p_0)+p_0^2-4p_0\right\}/(4\lambda^3 g),$$

   where $p_0=1+\lambda v_0^{2/3}$.

   (c) Show that, if $v_0=8ms^{-1}$, $\lambda^{\frac{3}{2}}=(1/8)sm^{-1}$, $N=3$, $g=(48/5)ms^{-2}$, the distance travelled is $5\ln(4/e)$ metres.

2. A point particle of mass m moves under gravity on a curve γ, which lies in a vertical plane, with s denoting distance along γ and $\psi(s)$ the (variable) angle of inclination from the horizontal at each position along γ.

   (a) Show that the tangential and normal components of velocity are $\dot{s}, 0$ respectively and those of acceleration are $\ddot{s}$, $\dot{s}^2/\rho$ respectively, where $\rho = ds/d\psi$.

   (b) If the reaction force R is normal to the curve, justify the governing equations $\ddot{s}=-g\sin\psi$, $m\dot{s}^2/\rho = R - mg\cos\psi$ for the motion.

   (c) If, also, γ is defined by $s^2=\sin\psi$, with $0<\psi<\pi/2$, and initially $s=s_0, \dot{s}=0$, solve the governing equations to give $\dot{s}^2$ as a function of s only and R as a function of ψ only.

3. Write down the radial and transverse components of acceleration in terms of the polar coordinates r and θ for a particle moving in a plane. If a particle of mass m is moving under the action of a force of magnitude $m\mu/r^n$ directed towards the origin, show that

$$\ddot{r} - \frac{h^2}{r^3} = -\frac{\mu}{r^n}, \quad h = r^2\dot{\theta}.$$

Show that a circular orbit of radius c is possible, and find the values of h and $\dot{\theta}$ in this case.

If the orbit of the particle is slightly disturbed so that $r = c + \rho$, where ρ is small, and if h is unaltered, find the differential equation for ρ, neglecting all terms of order ρ^2. Deduce that the orbit is stable when $n < 3$.

4. A particle of unit mass moves on the outside surface $z = K(r)$ of a smooth axisymmetric body, where r measures distance from the axis of symmetry (which is vertical) and z is distance measured upwards along the axis. Using conservation of energy and angular momentum, or otherwise, derive the governing equations

$$\tfrac{1}{2}\left\{[1 + K'(r)^2]\left(\frac{dr}{dt}\right)^2 + \frac{h^2}{r^2}\right\} + gK(r) = \text{constant},$$

$$r^2\frac{d\theta}{dt} = h = \text{constant},$$

with t, θ denoting time and azimuthal angle respectively.

Applying the equations to a cone of semi-vertical angle $\alpha (< \pi/2)$, with vertex uppermost, show that

$$\frac{d^2r}{dt^2} = \left(\frac{h^2}{r^3} + g\cot\alpha\right)\sin^2\alpha.$$

Deduce that the reaction R between the particle and the cone is given by

$$R = \left(g\tan\alpha - h^2/r^3\right)\cos\alpha.$$

5. A heavy body falls vertically through a cloud of particles at rest and accumulates particles at a rate kv (units of mass per unit time) when the body speed is v; here k is a constant. The body is initially at rest and of mass M.

   (a) Show that after falling a distance x the body has mass $m = M + kx$.

   (b) Deduce that the speed v satisfies

$$(M + kx)v\frac{dv}{dx} + kv^2 = (M + kx)g.$$

   (c) Hence or otherwise find v^2 as a function of x, g, k and M.

6. The tension T in an elastic string AB of negligible mass is given by $T = \lambda(\ell_1 - \ell)$, where ℓ is its natural length, ℓ_1 is its stretched length and λ is a stiffness constant. A particle of mass m_1 is attached to the end B and the end A is fixed. A second string BC of identical natural length and stiffness constant to AB, but with a particle of mass m_2 at the end C, is attached to the first particle at B.

   (a) Determine the equilibrium lengths of the two strings when the system hangs vertically under gravity.

   (b) Show that when the particles at B, C are subject to displacements x, y respectively, from equilibrium, their equations of motion are

$$\begin{aligned} m_1\ddot{x} &= \lambda(y - 2x), \\ m_2\ddot{y} &= \lambda(x - y). \end{aligned}$$

   (c) Deduce that motions are possible in which $x, y \propto \cos\omega t$, and find two possible values for ω^2.