UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M13B: Applied Mathematics 2

COURSE CODE : **MATHM13B**

UNIT VALUE : **0.50**

DATE : **04-MAY-05**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

05-C0958-3-120

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. Show that the tangential and normal components of acceleration of a point particle moving in a plane are $\ddot{s}$ and $\dot{s}^2/\rho$, where $\rho = \frac{ds}{d\psi}$ is the radius of curvature of the trajectory at the position of the particle and s is the distance along the curve from a fixed point on the trajectory; ψ is the tangent angle.

   A particle P moves on a plane curve γ with its speed proportional to the time, say $\dot{s} = Kt$ for some constant K. If (also) the acceleration of P makes a constant angle of 45° with the tangent, show that $\rho\ddot{s} = \pm\dot{s}^2$ and deduce that $\dot{\psi} = \pm 1/t$. Hence show that the intrinsic equation for γ is

$$s = ae^{\pm 2\psi} + b,$$

   where a, b are constants.

2. A small body, initially at rest at the top of a fixed sphere (of radius a), slides down the smooth outside surface in a vertical plane. Show that it departs from the surface when it has fallen through a vertical distance $a/3$, and that its horizontal component of velocity then is $(8ag/27)^{\frac{1}{2}}$.

   Find, for the subsequent (projectile) motion, expressions for the horizontal and vertical velocity components of the body in terms of time measured from the instant of departure.

3. A particle P (mass m) on a smooth horizontal table is joined by a light string (length l), passing through a smooth hole O in the table, to a second particle Q (mass m'). Initially P is projected horizontally, perpendicular to OP ($= a < l$), with speed V, and thereafter the string stays taut, with OQ vertical and of length z.

   (a) Derive the three equations $m(\ddot{r} - r\dot{\theta}^2) = m'(\ddot{z} - g)$, $r^2\dot{\theta} = h$ and $z + r = l$, where r is the length of OP and h is constant.

   (b) Deduce that r satisfies

$$(m + m')\ddot{r} - mh^2r^{-3} = -m'g.$$

   (c) Show that circular motion, $r = a$, is possible for a certain V, and find that V.

   (d) Show that, for small perturbations about the circular motion in (c) with fixed h,

$$(m + m')\ddot{\tilde{r}} + 3mh^2a^{-4}\tilde{r} = 0,$$

   where $r = a + \tilde{r}$.

4. A funnel has a smooth surface, given by $z = b(b/r)^n$ in cylindrical polar coordinates $z(> 0), r, \theta$ with the z-axis vertically downwards. Here $n(> 1)$ and $b(> 0)$ are constants. A particle of mass m is projected horizontally with speed u along the inner surface, at the level $z = b$.

By considering angular momentum and energy, or otherwise, show that

$$r^2\dot{\theta} = ub,$$

$$\left[1 + n^2\left(\frac{b}{r}\right)^{2n+2}\right]\dot{r}^2 + \left(\frac{ub}{r}\right)^2 - 2gb\left(\frac{b}{r}\right)^n = u^2 - 2gb,$$

where g denotes gravity.

If the particle is found to be moving horizontally again at the level $z = 2^n b$, prove that $3u^2 = (2^n - 1)2gb$.

5. A particle moving at speed v along a straight line collides with a stationary particle of equal mass. Show that their respective velocities just after collision are $(1 - \hat{e})v/2, (1 + \hat{e})v/2$ where $\hat{e}$ is the coefficient of restitution.

Three snooker balls A, B, C, of equal mass, lie in order on a straight line. Balls B, C are initially at rest and A has speed U directly towards B. If $\hat{e} = 0.6$ at each impact, and there are no external forces, show that

(i) after the first impact A, B have respective speeds $U/5$, $4U/5$;

(ii) after the second impact B, C have respective speeds $4U/25$, $16U/25$.

Find the speeds of A, B, C after the third impact.

6. (a) (i) Write down the displacement from its mean position of a simple harmonic oscillator with an amplitude A and a circular frequency Ω, the displacement having its largest negative value when $t = 0$.

(ii) The rise and fall of a tide in a harbour may be taken to be simple harmonic, the time interval between successive low tides being 12 hours 30 minutes. The harbour entrance has a depth of 4 metres at low tide and 10 metres at high tide. If the low tide occurs at noon, find the earliest time thereafter that a ship, needing 8.5 metres of water depth, can pass through the entrance.

(b) (i) Describe the phenomenon of beats. Given that

$$x_L(t) = A\cos(\Omega_L t - \epsilon_L), \qquad x_S(t) = a\cos(\Omega_S t - \epsilon_S),$$

$$\text{with} \quad \Omega_S - \Omega_L = \omega(\ll \Omega_L),$$

show that if

$$x(t) = x_L(t) + x_S(t) = C\cos(\Omega_L t - \epsilon_L + \theta) \quad \text{and} \quad \phi = (\epsilon_L - \epsilon_S) + \omega t,$$

then

$$C^2 = A^2 + a^2 + 2aA\cos\phi \quad \text{and} \quad \tan\theta = \frac{a\sin\phi}{A + a\cos\phi}.$$

(ii) A better approximation to the rise and fall of tides to that in (a) is to take the variation from the mean to be the sum of lunar oscillations, $x_L(t)$, and solar oscillations $x_S(t)$. The lunar oscillations have an amplitude 2.5 times the solar oscillations. If the harbour entrance has a depth of 10 metres at high water and 4 metres at low water during neap tides, what are the corresponding depths of the spring tides?
[Note: the neap tidal season occurs when the water displacement from the mean has the least amplitude; the spring tidal season occurs when the displacement has the greatest amplitude.]