UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M13B: Applied Mathematics 2

COURSE CODE	:	**MATHM13B**
UNIT VALUE	:	**0.50**
DATE	:	**19-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

04-C0952-3-90

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. A particle of mass m moves on a fixed line Ox under a resistance of magnitude mkv^2 and a restoring force $mf(x)$, where x is its displacement from the fixed point O and $v = \dot{x}$ is its velocity. The particle is projected from O with speed u, comes to rest at $x = a > 0$ and then returns to O with speed u_1.

   (a) Show that the governing equation for the outward motion can be written

   $$\frac{d}{dx}(e^{2kx}v^2) = -2e^{2kx}f(x).$$

   (b) Hence, or otherwise, show that

   $$u^2 = 2\int_0^a f(x)e^{2kx}\,dx.$$

   (c) Similarly, show that

   $$u_1^2 = 2\int_0^a f(x)e^{-2kx}\,dx.$$

2. The position of a point on a plane curve is given by $\mathbf{r}(s)$, where s is the arc length. If $\mathbf{r} = c\sinh^{-1}(s/c)\mathbf{i} + (s^2 + c^2)^{1/2}\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are constant orthogonal unit vectors and c is a constant, and if $\mathbf{t}$ $(= d\mathbf{r}/ds)$ is the unit vector along the tangent to the curve, find $\mathbf{t}$ in terms of s. Calculate $d\mathbf{t}/ds$ and hence find the unit normal $\mathbf{n}$ and the curvature κ in terms of s.

   A particle of mass m is moving along this curve, and its position at time t is given by $s(t)$. Find the velocity $(d\mathbf{r}/dt)$ and acceleration of the particle at any time t in terms of s and its derivatives. If the speed of the particle is constant and equal to u, find the force acting on the particle.

3. A particle P moves in a plane under the action of a force $f(r)$ per unit mass directed *away* from the fixed point O, where r is the length of the line OP. Assuming that the path of P does not pass through O, obtain the relations

$$r^2\dot{\theta} = h, \qquad \frac{d^2u}{d\theta^2} + u = \frac{-f(r)}{h^2u^2},$$

where h is a constant, $u = 1/r$ and the angle θ is referred to a fixed line through O in the plane of motion.

If $f(r) = k/r^2$, where $k > 0$ is a constant, and the initial conditions are $r = d$, $\dot{r} = 0$ and $\dot{\theta} = (k/d^3)^{1/2}$ at $\theta = 0$, show that the equation of the path is

$$\frac{d}{r} = 2\cos\theta - 1.$$

Show also that the speed of the particle tends to the value $(3k/d)^{1/2}$ as r tends to infinity.

4. A particle is moving on the inside of the surface $az = r^2$, where a is a constant and (r, z, θ) are cylindrical polar coordinates with z vertical, r radial, θ azimuthal. The surface is smooth and gravity g acts in the negative z direction. The particle initially is projected horizontally, with speed U, from a point on the surface where $z = 4a$.

(i) If in the subsequent motion the particle always lies between the planes $z = a$ and $z = 4a$, show (using conservation of energy and angular momentum, or otherwise) that $U^2 = 2ga$.

(ii) Find the magnitude of the radial velocity component as the particle crosses the plane $z = 9a/4$.

5. The mass of a spacecraft at time t is $m(t)$ and its velocity is $\mathbf{V}(t)$. For $t < 0$, $m(t) = M$ and $\mathbf{V} = U\mathbf{i}$ where M and U are constants and $\mathbf{i}$ is a constant unit vector. For $0 < t < T$, the craft encounters a stream of particles which have velocity $w(\cos\alpha\mathbf{i} + \sin\alpha\mathbf{j})$ where w and α are constants and $\mathbf{j}$ is a unit vector orthogonal to $\mathbf{i}$. A constant mass ρ of the particles enter the craft per unit time and these particles are thereafter stationary relative to the craft.

(a) If $\mathbf{V} = u\mathbf{i} + v\mathbf{j}$, show that

$$m\frac{du}{dt} + \frac{dm}{dt}u = \rho w\cos\alpha, \qquad m\frac{dv}{dt} + \frac{dm}{dt}v = \rho w\sin\alpha, \qquad \frac{dm}{dt} = \rho.$$

(b) Solve these equations to show that at time T the direction of motion of the craft has been turned through an angle β, where

$$\tan\beta = \frac{\rho w T\sin\alpha}{MU + \rho w T\cos\alpha}.$$

6. The springs of a car of mass M give it a period when empty of τ seconds, for small vertical oscillations about the equilibrium. These oscillations may be assumed to be simple harmonic.

(i) By making a suitable choice of the direction and origin of y, show that the governing equation for the oscillations is $M\ddot{y} = Mg - \lambda y$, and show how to relate λ to τ and M.

(ii) How far does the car sink (in equilibrium) when the driver and three passengers, each of mass m, sit in the car; and what is the new period of small oscillations about the mean?