UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M13B: Applied Mathematics 2

COURSE CODE : **MATHM13B**

UNIT VALUE : **0.50**

DATE : **02-MAY-03**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

03-C0945-3-90

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. A cycloid is described by the parametric equations

$$x = a(2\psi + \sin 2\psi) \qquad y = a(1 - \cos 2\psi),$$

where $-\frac{1}{2}\pi \leqslant \psi \leqslant \frac{1}{2}\pi$, the x-axis is horizontal and the y-axis is vertically upwards.

(a) Show that the distance s from the lowest point is given by

$$s = 4a \sin\psi.$$

(b) A particle of mass m moves under gravity g without friction on the cycloid. If the speed of the particle at the lowest point is $2(ga)^{\frac{1}{2}} \sin\alpha$, show that the particle oscillates between the points on the wire given by $\psi = \pm\alpha$. Show also that the reaction of the wire on the particle is

$$mg(2\cos\psi - \cos^2\alpha \sec\psi).$$

2. An explosion at a point $x = 0$ on a plane ($y = x\tan\beta$) inclined at an angle β to the horizontal x-axis hurls projectiles in all directions with speed u. Consider the motion in two dimensions (x, y) only.

(a) Show that, for a typical angle of projection θ to the horizontal, the projectile hits the plane at

$$x = u^2 \{\sin(2\theta - \beta) - \sin\beta\} / (g\cos\beta).$$

(b) Find the maximum range up the slope and the corresponding angle of projection.

3. Write down the radial and transverse components of acceleration in plane polar coordinates r, θ, and deduce that for a central force $r^2\dot{\theta} = h$ is constant. Show that the substitution $r = u^{-1}$ transforms $\ddot{r}$ into $-h^2u^2 d^2u/d\theta^2$. Hence or otherwise derive the differential equation

$$\frac{d^2u}{d\theta^2} + u = -\frac{f}{h^2u^2},$$

for a particle of unit mass under a central force $f(r)\hat{\mathbf{r}}$.

If $f(r) = kr^{-3}$, where k is a constant, find $u(\theta)$ given that the particle is projected from the point $r = a,\ \theta = 0$ with radial and transverse velocities U, V respectively. Show that the particle moves off to infinity when $\tan(q\theta) \rightarrow qV/U$, where $q^2 = (1 + ka^{-2}V^{-2})$.

4. A circular cone of semiangle $\frac{1}{4}\pi$ is placed with its axis vertical and vertex downwards, so that its surface is given in terms of cylindrical polar coordinates r, θ and z by $z = r$, $z > 0$. A particle of mass m is moving on the inner smooth surface of the cone under uniform gravity g. Show that $h = r^2\dot{\theta}$ is a constant and that the energy equation is

$$\dot{z}^2 + \frac{h^2}{2z^2} + gz = \frac{E}{m}.$$

If the particle is initially at $z = a$ and has velocity u in the horizontal direction, where $u^2 = 8ag/3$, find the bounds on the possible values of z in the motion.

Is it possible to choose u so that the particle remains at the level $z = a$ throughout the motion?

5. (a) Particles A and B, of masses m and $2m$ respectively, are travelling with velocities $\mathbf{i} + 3\mathbf{j}$ and $-2\mathbf{i} - \mathbf{j}$ respectively. The particles collide. After the collision, A is travelling at velocity $-\mathbf{i} + \mathbf{j}$. Find the velocity of B and the kinetic energy lost in the collision.

(b) Three spheres A, B and C have masses m_1, m_2 and m_3 respectively. They lie at rest on a smooth horizontal table, with B between A and C. A is projected towards B with speed u. After the collision B collides with C. All collisions are perfectly elastic. Show that if there is a third collision then

$$m_2(m_1 + m_2 + m_3) < 3m_1m_3.$$

6. Two waves travelling in the positive x-direction are given by $y_1 = \cos(k_1x - \omega_1t)$, $y_2 = \cos(k_2x - \omega_2t)$. Give an expression for the combined wave $y_3 = y_1 + y_2$ in terms of $\epsilon, k, \delta, \omega$, where

$$2\epsilon = k_2 - k_1, \quad 2k = k_1 + k_2, \qquad 2\delta = \omega_2 - \omega_1, \quad 2\omega = \omega_1 + \omega_2.$$

Sketch y_3 as a function of x, at a fixed time t, when $|\epsilon| \ll |k|$ and $|\delta| \ll |\omega|$, and give the speed of (a) individual crests, (b) the envelope.

If $\omega = F(k)$ for a given function F, show that the envelope (b) translates with speed dF/dk. Briefly describe the case $\omega = k^2$.

END OF PAPER