UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics M13B: Applied Mathematics 2

COURSE CODE	:	**MATHM13B**
UNIT VALUE	:	**0.50**
DATE	:	**14-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0940-3-80

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. A bead of unit mass is sliding down a straight wire, inclined at an angle α to the horizontal. The contact is rough, such that the tangential reaction opposing the motion is t multiplied by the normal reaction. Here t denotes time.

   (a) Show that, if the coordinate y is vertically upward, the bead motion is given by $y = x \tan \alpha$ and

$$\begin{aligned} \ddot{x} &= (t \cos \alpha - \sin \alpha) N, \\ \ddot{y} &= (t \sin \alpha + \cos \alpha) N - g, \end{aligned}$$

   where $N(t) > 0$ is unknown.

   (b) Deduce, from the equations in (a), or otherwise, that $N(t) = g \cos \alpha$.

   (c) Solve for $x(t)$, given that the bead is released from rest at $x = 0$ when $t = 0$. Then show that the bead comes to rest at a horizontal distance $2g \sin^3 \alpha / (3 \cos \alpha)$ from the origin.

2. A point particle of mass m moves under gravity on a curve γ, which lies in a vertical plane, with s denoting distance along γ and $\psi(s)$ the (variable) angle of inclination from the horizontal at each position along γ.

   (i) Show that the tangential and normal components of velocity are $\dot{s}, 0$ respectively and those of acceleration are $\ddot{s}$, $\dot{s}^2/\rho$ respectively, where $\rho = ds/d\psi$.

   (ii) If the reaction force R is normal to the curve, justify the governing equations $\ddot{s} = -g \sin \psi$, $m\dot{s}^2/\rho = R - mg \cos \psi$ for the motion.

   (iii) If, also, γ is defined by $s^2 = \sin \psi$, with $0 < \psi < \pi/2$, and initially $s = s_0, \dot{s} = 0$, solve the governing equations to give $\dot{s}^2$ as a function of s only and R as a function of ψ only.

3. The position of a particle, moving in a plane, has polar coordinates r, θ, with corresponding unit vectors $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}$, so that its position vector is $\mathbf{r} = r\hat{\mathbf{r}}$. Show that the acceleration of the particle is

$$(\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\hat{\boldsymbol{\theta}}.$$

You may assume the rates of change (with respect to t) of $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}$ to be $\dot{\theta}\hat{\boldsymbol{\theta}}$, $-\dot{\theta}\hat{\mathbf{r}}$ respectively.

The particle is moving under the action of an unspecified radial force directed towards the origin O. Initially the particle is at $r = a, \theta = 0$, with velocity $u\hat{\boldsymbol{\theta}}$.

(i) Deduce that $r^2\dot{\theta} = ua$.

(ii) If the orbit of the particle is a closed curve enclosing an area A, show that the time T for a single circuit is given by $T = 2A/ua$.

(iii) Show that O cannot lie outside the orbit.

4. The surface of a smooth funnel is given by $z = a^4 r^{-3}$, in cylindrical polar coordinates r, θ, z, with the z-axis vertically downwards. A particle of mass m is projected horizontally with speed u, along the inner surface, at the level $z = a$.

By considering angular momentum and energy, or otherwise, show that $r^2\dot{\theta} = ua$ and

$$\left(1 + 9\frac{a^8}{r^8}\right)\dot{r}^2 + u^2\frac{a^2}{r^2} - 2g\frac{a^4}{r^3} = u^2 - 2ga,$$

where g denotes gravity.

If the particle is found to be moving horizontally again at the level $z = 8a$, prove that $u^2 = 14ga/3$.

5. A heavy body falls vertically through a cloud of particles at rest and accumulates particles at a rate kv (units of mass per unit time) when the body speed is v; here k is a constant. The body is initially at rest and of mass M.

   (a) Show that after falling a distance x the body has mass $m = M + kx$.

   (b) Deduce that the speed v satisfies

$$(M + kx)v\frac{dv}{dx} + kv^2 = (M + kx)g.$$

   (c) Hence or otherwise find v^2 as a function of x, g, k and M.

6. The tension T in an elastic string AB of negligible mass is given by $T = \lambda(\ell_1 - \ell)$, where ℓ is its natural length, ℓ_1 is its stretched length and λ is a stiffness constant. A particle of mass m_1 is attached to the end B and the end A is fixed. A second string BC of identical natural length and stiffness constant to AB, but with a particle of mass m_2 at the end C, is attached to the first particle at B.

   (a) Determine the equilibrium lengths of the two strings when the system hangs vertically under gravity.

   (b) Show that when the particles at B, C are subject to displacements x, y respectively, from equilibrium, their equations of motion are

$$\begin{aligned} m_1\ddot{x} &= \lambda(y - 2x), \\ m_2\ddot{y} &= \lambda(x - y). \end{aligned}$$

   (c) Deduce that motions are possible in which $x, y \propto \cos\omega t$, and find two possible values for ω^2.