UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc.   *M.Sci.*

Mathematics M13A: Applied Mathematics 1

COURSE CODE	:	**MATHM13A**
UNIT VALUE	:	**0.50**
DATE	:	**23-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Explain and justify the statement

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

and use it to show that

$$\mathbb{P}(A \mid \overline{B}) = \frac{\mathbb{P}(A) - \mathbb{P}(A \mid B)\mathbb{P}(B)}{1 - \mathbb{P}(B)}.$$

There are three people in a room, none of whom has a birthday in November or December. Assuming that the other months are all equally likely:

(a) What is the probability that their birthdays all fall in different months?

(b) What is the probability that none of their birthdays is in January?

(c) If their birth months are all different, what is the probability that one of them has a birthday in January?

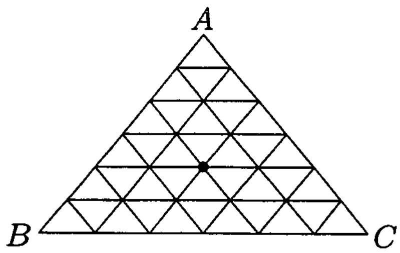
(d) What is the probability that at least one of them has a birthday in January if we are told that at least two of them share a birth month?

[Hint: for part (d), let A be the event that at least one person has a birthday in January, and B be the event that all three have different birth months.]

2. If p is the probability of success in each trial of a sequence of Bernoulli trials, show that the probability of exactly r successes in n trials is given by the binomial distribution

$$b(r; n, p) = \binom{n}{r} p^r (1-p)^{n-r}.$$

Three ants crawl on a triangular grid as shown:



The ant which starts from point A crawls downwards. Each edge it crawls along takes 1 second, and at each junction it is equally likely to head left (and down) or right (and down).

Show that the probability it reaches the central point • after 4 seconds is $3/8$.

The ant which starts from B at the same time crawls either directly right, or right and upwards. It also takes one second to cover each edge, and is equally likely to choose either of its directions at each junction. The ant at C behaves similarly, but moves either left or left and upwards.

Find the probability that at least two ants are at the central point after 4 seconds.

3. Consider a particle of unit mass moving under the action of a force which depends only on the position of the particle:

$$\ddot{x} = -\frac{\mathrm{d}V(x)}{\mathrm{d}x}.$$

Derive the energy equation for this motion.

Now consider a particle of unit mass which is released from a height $x = 3\beta^2/2$ at time $t = 0$ and falls under gravity (gravitational acceleration g). At what time will it reach the position $x = 0$?

When the particle reaches $x = 0$ it lands on a spring. If $x < 0$ the particle is pushed upwards with a force $-gx/\beta^2$.

Write down the equations governing the motion for both $x > 0$ and $x < 0$. Hence find a potential function for the entire motion. Sketch your potential function. Determine the energy for this system.

Where will the particle first come to rest? At what time will this occur? Show that the particle bounces (performs oscillations) with period

$$T = \left(2\sqrt{3} + \frac{4\pi}{3}\right)\frac{\beta}{\sqrt{g}}.$$

4. A car of mass m accelerates under the driving force:

$$F_1 = m(U - v)$$

where v is its speed and U is a positive constant. If, at time $t = 0$, it is at rest at the origin $x = 0$, find its speed and position as a function of time.

At time $t = 2$ the car brakes, applying a retardation force

$$F_2 = -mkv.$$

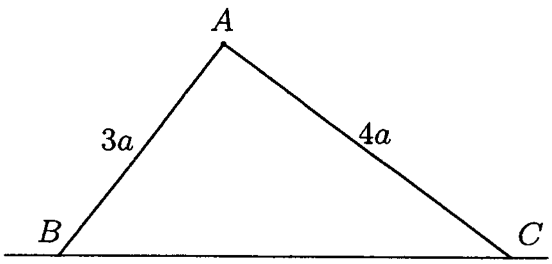
Where will the car come to a halt?

If, instead, the car brakes with a force

$$F_3 = -mkv^2$$

describe the subsequent motion.

5. Two uniform rods having equal weight W and lengths $3a$ and $4a$ respectively are linked by a smooth hinge at A. They are resting on a rough horizontal surface as shown so that the angle between them at A is a right angle.



The friction coefficient μ between each rod and the surface satisfies $\mu > 24/43$. If a weight is hung from the hinge at A, and gradually increased, which rod will slip first?

Show that the maximum weight which can be hung from the hinge at A without causing either of the rods to slip is

$$\frac{(43\mu - 24)}{(24 - 18\mu)}W.$$

6. A length of rope is lying in contact with a plane curve whose intrinsic coordinates are (s, ψ). The rope lies on top of the curve. The coefficient of friction between the curve and the rope is μ. The rope has weight W per unit length and is on the point of slipping in the direction of increasing s. Carry out a force balance on a small section of the rope between s and $s + \delta s$, both tangential and normal to the curve, and show that the tension T in the rope satisfies

$$\frac{\mathrm{d}T}{\mathrm{d}s} = W \sin\psi + \mu W \cos\psi - \mu T \frac{\mathrm{d}\psi}{\mathrm{d}s}.$$

Now suppose the curve is given by

$$y = a(1 - \cosh(x/a))$$

and the rope is in contact with the portion from the origin, $x = 0$, to the point C where $x = a \sinh^{-1} 1$. Find the intrinsic equation of the curve, and deduce a differential equation for $T(\psi)$. Show that your equation is satisfied by

$$T = ke^{-\mu\psi} - aW \sec\psi$$

for some constant k. If the tension at the origin is T_0, deduce the tension at C.