UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M13A: Applied Mathematics 1

COURSE CODE : **MATHM13A**

UNIT VALUE : **0.50**

DATE : **13-MAY-05**

TIME : **10.00**

TIME ALLOWED : **2 Hours**

05-C0957-3-110

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Explain the meaning of the binomial distribution, and give an equation for it, defining all the terms you use in the equation.

   What is the relationship between the binomial and Poisson distributions? Give the equation for the Poisson distribution, defining all the terms you use.

   (b) On average 0.2% of pens produced are defective. Suppose a batch produced contains 2000 pens.

   (i) What is the expected number (mean number) of defective pens produced?

   (ii) Give both an exact expression, and a simpler approximate expression, for the probability that exactly this number of defective pens is produced. You need not evaluate these expressions.

2. DNA is a molecule made of long sequences of *bases*. Each base is one of the letters

$$A, C, G, T.$$

   (a) How many different sequences of length n can be built?

   (b) Suppose I am given a randomly chosen DNA fragment of length n! What is the probability that:

   (i) it begins with A?

   (ii) it begins and ends with the same letter?

   (iii) no two consecutive letters are the same?

   (iv) it reads the same backwards as forwards?

   [Hint: in (iv) you will need to consider the cases of even and odd n separately.]

   (c) A continuous probability distribution has probability density function

$$f(x) = \begin{cases} 0 & x < 0 \\ \lambda \exp\left[-\lambda x\right] & x \geq 0. \end{cases}$$

   (i) For a given positive constant a, what is the probability that $x < a$?

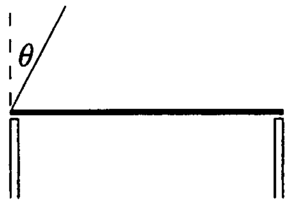
   (ii) What is the mean of the distribution?

3. (a) Three particles A, B and C at positions $\underline{r}_1$, $\underline{r}_2$ and $\underline{r}_3$ are connected by light springs. An external force $\underline{F}_1$ is applied to particle A, an external force $\underline{F}_2$ to particle B, and an external force $\underline{F}_3$ to particle C.

Show that if the forces are all in equilibrium then

$$\underline{F}_1 + \underline{F}_2 + \underline{F}_3 = \underline{0} \quad \text{and} \quad \underline{r}_1 \times \underline{F}_1 + \underline{r}_2 \times \underline{F}_2 + \underline{r}_3 \times \underline{F}_3 = \underline{0}.$$

(b) A uniform rod of weight W rests in equilibrium on two identical rough posts as shown in the diagram. The coefficient of friction between the rod and the posts is μ. A string is attached to one end of the rod, making an angle θ with the vertical.



Initially the string is slack, then the tension is gradually increased until equilibrium is broken. What are the two possible events that will break equilibrium? For what values of μ does each of them occur?

4. A particle of mass m is projected upwards from the point $x = 0$ with speed $v = v_0$ at time $t = 0$. Denote by g the acceleration due to gravity.

(a) Assume that the only force acting on the particle is that due to gravity. Write down the equation governing the motion of the particle. Find the speed and position of the particle as a function of time.

What is the maximum height reached by the particle? Show that the particle returns to the point $x = 0$ at a time $2v_0/g$.

(b) Now suppose instead that the particle is moving under the action of both gravity and air resistance, which is a retardation force of magnitude mkv. Write down the new equation governing the motion.

Obtain a relation between position, x, and speed, v. What is the maximum height reached by the particle? If k is small relative to g/v_0, show that the height reached is reduced by an amount $kv_0^3/3g^2 + O(k^2)$ by air resistance.

You may assume without derivation the Taylor series:

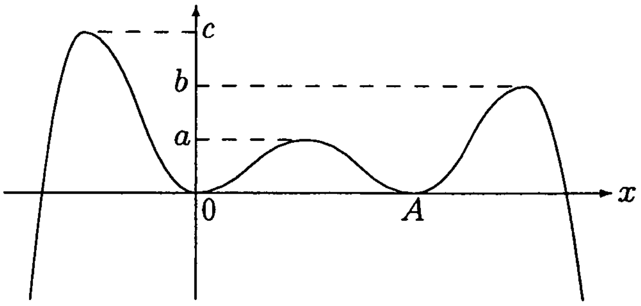
$$\ln(1+\alpha) = \alpha - \alpha^2/2 + \alpha^3/3 + O(\alpha^4).$$

5. Suppose a particle of mass m is moving in one dimension under the action of a force $-m\,dV/dx$ so that

$$\ddot{x} = -\frac{dV}{dx}.$$

(a) Derive the energy equation for the particle.

Now suppose the function $V(x)$ is as shown:

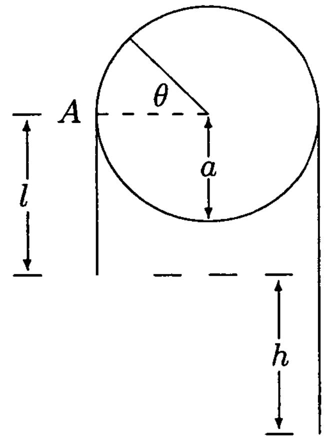


Initially, the particle is projected leftwards from the point $x = 0$ with speed v_0.

(b) Describe the motion in qualitative terms for different values of v_0. Give ranges of v_0^2 for which each type of motion occurs.

(c) Sketch the phase plane for this system. Include the possibility of particles starting at points other than $x = 0$. Label a trajectory corresponding to each type of motion you have described in (b).

6. Consider a heavy chain of weight W per unit length, hanging over a fixed rough cylinder of radius a as shown in the diagram.



The coefficient of friction between the chain and the cylinder is μ, and the chain is in limiting equilibrium, on the point of slipping to the right. The angle θ is as shown on the diagram, increasing from 0 at A to π at B.

(a) What is the tension at the point A, in terms of W, l and h? What is the tension at the point B, in terms of W, l and h?

(b) Give the intrinsic equation of the plane curve described by the chain between A and B, taking $s = 0$ at the point A. Now express both s and ψ in terms of θ.

(c) Consider the portion of chain of length $a\delta\theta$ between θ and $\theta+\delta\theta$, where $\delta\theta \ll 1$. Carry out a force balance and let $\delta\theta \to 0$ to show that, if the normal reaction from the cylinder on the chain is R per unit length, and the tension in the chain is T, then

$$aR = aW\cos\theta + T \quad \text{and} \quad \mu aR + aW\sin\theta = \frac{\mathrm{d}T}{\mathrm{d}\theta}.$$

(d) Eliminate R to obtain a first-order differential equation for T, and find the general solution of this equation. Using the values of tension obtained in (a), deduce the height difference h.