UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. M.Sci.

Mathematics M13A: Applied Mathematics 1

COURSE CODE	:	**MATHM13A**
UNIT VALUE	:	**0.50**
DATE	:	**13-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0944-3-100

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. A random number table consists of a succession of digits each chosen at random and independently from the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

   (a) Two successive digits are taken from such a table. Show that the probability that their sum is 9 is $1/10$.

   (b) Four successive digits are taken from the table. Calculate the probability that the sum of the first two equals the sum of the third and fourth.
   [You may wish to use the formula $\sum_{r=1}^{n} r^2 = n(n+1)(2n+1)/6$.]

2. Use the binomial theorem to show that

$$\sum_{m=0}^{n} {}^nC_m p^m q^{n-m} = 1 \quad , \quad \sum_{m=0}^{n} m \, {}^nC_m p^m q^{n-m} = np$$

where $p + q = 1$.

A man is walking along a road. He tosses a biased coin which shows a head with probability $2/3$. If the coin shows a head he moves three paces forwards, and if it shows a tail he moves one pace backwards. Show that, after n moves, the probability that he has reached $4m - n$ paces ahead of his starting point is

$${}^nC_m 2^m 3^{-n}.$$

Show that, after 10 moves, the probability that he will be behind his starting point is 201×3^{-10}.

Show that, after n moves, his expected position will be $5n/3$ paces forward.

[If the N possible outcomes of an experiment are given numerical values $m_1, m_2, \ldots, m_N$ and if they occur with probabilities $p_1, p_2, \ldots, p_N$ the expected value is $\sum_{i=1}^{N} m_i p_i$.]

3. A body is acted on by a system of n forces $\mathbf{F}_i$ acting at points with position vector $\mathbf{r}_i$. Write down the definition of the resultant force $\mathbf{F}$ and the resultant moment $\mathbf{G}_0$ about the origin. Show that, if $\mathbf{F} = 0$, the resultant moment is independent of the point about which it is taken.

A light ladder of length L stands on a rough floor and leans against a rough wall, making an acute angle θ with the vertical wall. A man climbs the ladder to a vertical height h above the ground. If the ladder is on the point of slipping at both ends, and the coefficient of friction between the ladder and both the ground and the wall is μ, show that

$$\frac{h}{L} = \mu \frac{(\mu \sin\theta + \cos\theta)}{(1+\mu^2)\tan\theta}.$$

If the man has reached the mid-point of the ladder when it is about to slip show that $\mu = \tan \frac{1}{2}\theta$.

4. A uniform chain hangs under its own weight. Consider the equilibrium of a length AB of this chain, where B is the lowest point of the chain, at which the tangent is horizontal. Let the intrinsic coordinates of A be (s, ψ) and the tension at A be T, while at B the coordinates are $(0, 0)$ and the tension is T_0. Show that

$$T = T_0 \sec\psi \text{ and } s = c\tan\psi$$

where $c = T_0/w$, w being the weight per unit length.

Write $\dfrac{dy}{d\psi} = \dfrac{dy}{ds}\dfrac{ds}{d\psi}$ and integrate to find the vertical coordinate y as a function of ψ, choosing $y = c$ when $\psi = 0$. Similarly find the horizontal coordinate x as a function of ψ, choosing $x = 0$ when $\psi = 0$. Finally, eliminate ψ to obtain the equation of the curve as $y = c\cosh(x/c)$ with $s = c\sinh(x/c)$.

The end links of a uniform chain can slide on a fixed rough horizontal rod. Show that the ratio of the extreme span to the length of the chain is

$$\mu \sinh^{-1}\left(\frac{1}{\mu}\right)$$

where μ is the coefficient of friction.

5. A particle of unit mass can move along the x-axis under the action of a force with potential $V(x)$ where

$$V(x) = U^2 \frac{ax}{a^2 + x^2}$$

and U and a are positive constants. Show that the only stable equilibrium point is at $x = -a$.

The particle is placed at $x = -a$ and given a speed λU in the positive x direction. If $0 < \lambda < 1$ show that the particle will oscillate between the two points $x = x_1$ and $x = x_2$ where x_1 and x_2 are the roots of the equation

$$(1 - \lambda^2)(x^2 + a^2) + 2ax = 0.$$

Describe the motion of the particle if, instead,

(i) $1 < \lambda < 2^{\frac{1}{2}}$

(ii) $\lambda > 2^{\frac{1}{2}}$.

6. A particle of mass m moves in one dimension under an attractive force γ/x^2 towards the origin ($x = 0$) and against a resistance $k\dot{x}^2/x$. Write down the equations of motion for the cases

(i) $x > 0$, $\dot{x} > 0$,

(ii) $x > 0$, $\dot{x} < 0$.

Both γ and k are positive constants.

Define $v = \dot{x}$, and $T(x) = \frac{1}{2}mv^2$ to be the kinetic energy, and show that the equations of motion may be written as

$$\frac{dT}{dx} \pm \frac{2k}{m}\frac{T}{x} = -\frac{\gamma}{x^2}.$$

If $2k > m$, show that, under the initial condition $\dot{x} = u > 0$ where $x = a > 0$,

$$T(x) = \left(\frac{a}{x}\right)^{2k/m} \left\{\frac{1}{2}mu^2 + \frac{\gamma}{\left(\frac{2k}{m} - 1\right)a}\right\} - \frac{\gamma}{\left(\frac{2k}{m} - 1\right)x}$$

until $\dot{x}$ becomes zero.

Find the solution for $T(x)$ under the same initial conditions when $2k = m$.

END OF PAPER