UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics M13A: Applied Mathematics 1

COURSE CODE	:	**MATHM13A**
UNIT VALUE	:	**0.50**
DATE	:	**16-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0939-3-90

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) A box contains five red and three green balls. One ball is drawn at random. It is replaced and two more balls of this same colour are added to the box. A second ball is then drawn.

   (i) Find the probability of a red ball on the second draw.

   (ii) Find the probability of one ball of each colour in the two draws.

   (iii) If the second ball drawn is red, find the probability that the first ball drawn was red.

   (b) The probability that Jim wins at table tennis is a function of his previous performance. The probability that he wins a given game is $\left(\frac{1}{2}\right)^{k+1}$ where k is the number of games that he has won so far. If he plays three games, what is the probability that he will win at least two?

2. If p is the probability of success in each trial of a sequence of Bernoulli trials, show that the probability of exactly r successes in n trials is given by the binomial distribution

$$^{n}C_{r}\, p^{r}(1-p)^{n-r},$$

and find its mean μ.

A square city has $2N+1$ roads running due North and $2N+1$ roads running due East so that, measured from the centre of the city, each intersection can be described by the coordinates (m,n) with $-N \leqslant m \leqslant N$ and $-N \leqslant n \leqslant N$. A person P_1 starts from $(-N,N)$ and moves in a Northerly or Easterly direction, choosing either direction at each intersection with equal probability. Show that the probability that he will reach the city centre is

$$Q=\frac{(2N)!}{(N!)^2 2^{2N}}.$$

Persons P_2, P_3 and P_4 start from the other corners of the city at the same time as P_1 starts, and they all walk at the same speed. Person P_2 starts from the North-West corner and moves either Southerly or Easterly, P_3 from the North-East corner and moves either Southerly or Westerly, and P_4 starts from the South-East corner and moves either Northerly or Westerly. Show that the probability that at least two of them will meet at the centre is

$$Q^2(3Q^2-8Q+6).$$

3. A set of forces $\mathbf{F}_i$ acting at points P_i $(i = 1, 2, ...)$ has vector sum $\mathbf{F}$ and sum of vector moments $\mathbf{G}$ about a point O. Show that the scalar product $\mathbf{F}.\mathbf{G}$ is the same for all points O. Show also that the forces are equivalent to a single force if $\mathbf{F}.\mathbf{G} = 0$ and $\mathbf{F} \neq 0$.

   Forces with components $(F_0, 0, 0)$, $(0, F_0, 0)$ and $(0, 0, F_0)$ act at the points $(0, 0, 0)$, $(a, 0, c)$ and $(0, b, 0)$ respectively. Show that the forces are equivalent to a single force if $c = a + b$, and in that case find where the line of action of this force intersects the plane $z = 0$.

4. A length of rope is in contact with a rough plane curve. The coefficient of friction between them is μ and the weight of the rope may be neglected. Consider the limiting equilibrium of an element AB of rope that is about to slip in the direction from A to B. Suppose that at A, B the tangents to the curve make intrinsic angles $\psi \mp \delta\psi/2$ respectively with a fixed direction, and that the tensions there are $T \mp \delta T/2$. By resolving along and perpendicular to the tangent to the element at the point where the intrinsic angle is ψ, show that $dT/d\psi = \mu T$, and hence that

$$T_B = T_A \exp[\mu(\psi_B - \psi_A)].$$

   Two weights P and Q hang in equilibrium from a string which passes over a rough circular cylinder in a plane perpendicular to the axis, which is horizontal. If P is on the point of descending, what weight may be added to Q without causing Q to descend?

5. A particle of mass m moves on a fixed line Ox under a resistance of magnitude mkv^2 and a restoring force $mf(x)$ towards the origin O. Here x is the displacement of the particle from O and $v = \dot{x}$ is its velocity. The particle is projected from O with speed u, comes to rest at $x = a > 0$, and returns to O with speed u_1. Show that

$$u^2 = 2\int_0^a f(x)e^{2kx}dx \quad \text{and} \quad u_1^2 = 2\int_0^a f(x)e^{-2kx}dx.$$

   If the restoring force is a constant of magnitude Ak, show that

$$u_1^2 = A(1 - e^{-2ka})$$

   and that the particle next comes to rest at the negative value of x given by

$$e^{-2kx} = 2 - e^{-2ka}.$$

6. A particle of mass m is moving under the action of a force with potential

$$V(x) = V_0 \frac{x^2(x^2-3)}{4-x^2}, \qquad |x| < 2, \qquad V_0 > 0.$$

Sketch the graph of $V(x)$ for $|x| < 2$, given that it has a minimum at both $x = \pm 2^{1/2}$, and a maximum at the origin.

If the total energy $E = 7V_0/2$, show that the particle moves between the positions

$$x = \pm \left(\frac{7}{2}\right)^{1/2},$$

and find its speed when $x = 0$.

Suppose that $E = 0$, and that at time T the particle is within a small distance of the origin and is moving towards it. Show that the energy equation may be approximated by

$$\dot{x}^2 = 3V_0 x^2/(2m).$$

Find x as a function of the time, and deduce that the particle will take an infinite time to reach the origin.