UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M233: Analytical Dynamics

COURSE CODE	:	**MATHM233**
UNIT VALUE	:	**0.50**
DATE	:	**03-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1048-3-60

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Let $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\hat{B} = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ be two sets of right-handed orthonormal vectors. Define the transition matrix H from $\hat{B}$ to B. Given the relations

$$\mathbf{e}_i = H_{ij}\hat{\mathbf{e}}_j, \quad \hat{\mathbf{e}}_i = H_{ji}\mathbf{e}_j,$$

prove that H is orthogonal. Show that the matrix $\Omega = \dot{H}H^T$ is skew-symmetric. If $\Omega_{jk} = \epsilon_{ijk}\omega_i$ give a physical interpretation of the vector $\boldsymbol{\omega} = \omega_i\mathbf{e}_i$.

(b) In an inertial frame of reference a particle of mass m moves in a horizontal (x, y)-plane and is subject to a force

$$\mathbf{F} = mf(r)\hat{\mathbf{r}} + m\alpha\mathbf{k} \times \dot{\mathbf{r}},$$

where $\mathbf{r}$ is the position vector of the particle, $r = |\mathbf{r}|$, $\hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}|$, $f(r)$ is an arbitrary function of r, α a given constant and $\mathbf{k}$ a unit vector normal to the (x, y)-plane.
Find a frame of reference rotating with angular velocity $\boldsymbol{\omega}$ (to be found) and $f(r)$ such that the force on the particle is zero.
According to such a rotating observer at $t = 0$ the particle is such that $r = 0$ and $\dot{r} = u$ where u is a positive constant. Describe the motion of the particle in the inertial frame of reference.

2. (a) Consider a system of N particles of constant masses $m_1, m_2, \ldots, m_N$ and position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$.

(i) Define the centre of mass $\mathbf{R}$. Show that the total kinetic energy T can be written as $T = T_{CM} + T_{rel}$, where T_{CM} is the kinetic energy of the centre of mass and T_{rel} is the kinetic energy about the centre of mass.

(ii) The particles are subject to external and internal forces which are derivable from potentials $V^{(e)}$ and V_{int} respectively. If $V^{(e)}$ is a function of the position of the centre of mass only, show that $E_{CM} = T_{CM} + V^{(e)}$ is constant and $E_{int} = T_{rel} + V_{int}$ is constant.

(b) Two particles of masses m_1 and m_2 are located in a uniform gravitational field and the vector joining them is $\mathbf{r}$. In addition to gravity, the particles experience a force of attraction of magnitude α/r^2, where $r = |\mathbf{r}|$ and α is constant. Show that the external potential is a function of the position of the centre of mass only and that

$$\mu|\dot{\mathbf{r}}|^2 - \frac{2\alpha}{r} = \text{const},$$

where $\mu = m_1 m_2/(m_1 + m_2)$.

3. (a) Write down Lagrange's equation of motion for a system having one generalized coordinate q and Lagrangian $L = L(q, \dot{q}, t)$. Let $\bar{L}$ be another Lagrangian defined by

$$\bar{L} = L + \frac{d}{dt} g(q, t),$$

where $g(q, t)$ is an arbitrary differentiable function. Show that $\bar{L}$ gives the same equation of motion as L.

(b) A pendulum consisting of a light rod AB of length a with a heavy mass m located at B is free to swing in the (x, z)-plane. The pivot end of the pendulum A is forced vertically to be a given distance $\gamma(t)$ from an origin O. There is a uniform gravitational field $-g\mathbf{k}$. Let θ be the angle between AB and the downward vertical. Show that the Lagrangian for the pendulum is

$$L = \frac{1}{2} m(a^2\dot{\theta}^2 + 2a\dot{\theta}\dot{\gamma}\sin\theta) + mga\cos\theta + h(t),$$

where $h(t)$ is a function of t only (to be found).

By considering $d(\dot{\gamma}\cos\theta)/dt$ and part (a), show that the Lagrangian

$$\bar{L} = \frac{1}{2} ma^2\dot{\theta}^2 + ma(g + \ddot{\gamma})\cos\theta,$$

gives rise to the same dynamics as L. Find the corresponding Hamiltonian $\bar{H}$ and the find the general form of $\gamma(t)$ so that $\bar{H}$ is constant. Comment briefly on the special case $\ddot{\gamma} = -g$.

4. (a) The dynamics of a system are governed by a Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$. Define the generalised momenta p_i and the Hamiltonian H of the system and derive Hamilton's equations of motion.

   (b) A system with two degrees of freedom has Hamiltonian

$$H(q_1, q_2, p_1, p_2) = q_1 p_1 - q_2 p_2 - a q_1^2 + b q_2^2,$$

   where a and b are constants. Show that

$$\begin{aligned} (i) \quad & (p_2 - b q_2)/q_1 = \text{constant}, \\ (ii) \quad & q_1 q_2 = \text{constant}, \\ (iii) \quad & \ln q_1 = t + \text{constant}. \end{aligned}$$

5. (a) A rigid body of density ρ and volume V rotates about a fixed point P. Show that its kinetic energy is

$$T = \frac{1}{2}\omega_i \omega_j J_{ij},$$

   where

$$J_{ij} = \int_V \rho(r_k r_k \delta_{ij} - r_i r_j)dV,$$

   is the ijth element of the inertia matrix at P in a rest frame of the body with axes $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, and ω_i is the ith component of the angular velocity of the body.

   If another set of axes $B' = \{\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ is chosen at P show that

$$J_{ij} = H_{ip} J'_{pq} H_{jq},$$

   where

$$J_{pq} = \int_V \rho(r'_k r'_k \delta_{ij} - r'_p r'_q)dV,$$

   and H is the transition matrix from B' to B. Hence explain why it is always possible to choose B' such that the inertia matrix is diagonal. What are the axes called in this case?

   (b) If the inertia matrix a rigid body is diagonal with $J_{11} = A$, $J_{22} = B$ and $J_{33} = C$ write down Euler's equations for the motion of the body for the case when there are no applied torques. The body has rotational symmetry about a particular axis. Show that the component of the angular velocity along this axis is constant. Show also that the magnitude of the angular velocity in the plane perpendicular to the symmetry axis is constant.

6. A symmetric top moves about a fixed point P in a uniform gravitational field. Explain what is meant by the angles of precession ϕ and nutation θ. The Lagrangian $L(\psi, \phi, \theta)$ for the top with moments of inertia A and C is, in the usual notation,

$$L = \frac{1}{2}A\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}C\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - mga\cos\theta$$

Derive three conservation laws for the motion of the top in the form of three equations. Give a physical interpretation for each of these equations.

A top is released with initial conditions $\theta = \pi/4$, $\dot{\theta} = \dot{\phi} = 0$ and $\dot{\psi} = \omega$. It is observed that the symmetry axis of the top reaches, but never falls below, the horizontal during its motion. Show that

$$C^2\omega^2 = 2\sqrt{2}mgaA.$$

Sketch the trajectory that the intersection of the symmetry axis of the top makes with the unit sphere.