UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M233: Analytical Dynamics

COURSE CODE	:	**MATHM233**
UNIT VALUE	:	**0.50**
DATE	:	**06-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

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TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Write down the expression for the Coriolis force acting on a particle of mass m moving with velocity $\dot{\mathbf{r}}$ measured by an observer rotating with angular velocity $\boldsymbol{\omega}$ relative to an inertial frame.

   Let a reference frame with origin O and Cartesian axes (x, y, z) be fixed relative to the surface of the rotating earth at co-latitude θ (i.e. $0 \leq \theta \leq \pi$, where $\theta = 0$ corresponds to the north pole). Here, increasing x is east, increasing y is north and increasing z is upwards (i.e. in the opposite direction to the effective gravity $\mathbf{g}^*$). The earth is assumed to rotate steadily with angular velocity $\boldsymbol{\omega}$.

   Briefly explain the difference between the effective gravity $\mathbf{g}^*$ and actual gravity $\mathbf{g}$. Find the components of $\boldsymbol{\omega}$ in this frame of reference. Show that the motion of a particle of mass m under gravity is governed by

$$\ddot{x} - 2\omega\dot{y}\cos\theta + 2\omega\dot{z}\sin\theta = 0,$$
$$\ddot{y} + 2\omega\dot{x}\cos\theta = 0,$$
$$\ddot{z} - 2\omega\dot{x}\sin\theta = -g^*,$$

   where $\omega = |\boldsymbol{\omega}|$ and $g^* = |\mathbf{g}^*|$. Assuming $\mathbf{g}^*$ and θ are constant, by integrating the second and third of these equations with respect to time and substituting into the first equation, show that

$$\ddot{x} + 4\omega^2 x = 2\omega(v_0\cos\theta - w_0\sin\theta) + 2\omega g^* t\sin\theta,$$

   where v_0 and w_0 are constants. Hence find the general solution for x.

   If a particle falls from rest at O, find x as a function of t. The particle falls only for a brief time before it hits the ground, so that ωt is small throughout its motion. Use a series expansion of solution for x in ωt to show, to leading order,

$$x = \frac{1}{3}g^*\omega t^3\sin\theta.$$

   Explain briefly how an inertial observer would account for this eastward deflection of the falling particle.

2. (a) Consider a system of N particles of constant masses $m_1, m_2, \ldots, m_N$ and position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$.

(i) Define the centre of mass $\mathbf{R}$.

(ii) Show that the total kinetic energy T can be written as $T = T_{CM} + T_{rel}$, where T_{CM} is the kinetic energy of the centre of mass and T_{rel} is the kinetic energy about the centre of mass.

(b) If the ith particle is subject to a uniform external force of the form $\mathbf{F}_i^{(e)} = m_i\mathbf{c}$, $i = 1, 2, \ldots, N$, where $\mathbf{c}$ is a constant vector, show that the total external torque about the centre of mass is zero.

(c) Two particles with masses m_1 and m_2 move in the (x, z)-plane and are connected by a light rigid rod of length l. A uniform external force acts on the particles. Take (x_{CM}, y_{CM}, θ) as generalised coordinates, where (x_{CM}, y_{CM}) is the position of the centre of mass and θ is the angle the rod makes with the x-axis. Explain why the potential V for the force does not depend on θ, and show that the Lagrangian is

$$L = \frac{M}{2}\left(\dot{x}_{CM}^2 + \dot{y}_{CM}^2\right) + \frac{\mu}{2} l^2 \dot{\theta}^2 - V(x_{CM}, y_{CM}),$$

where $M = m_1 + m_2$ and $\mu = m_1 m_2/(m_1 + m_2)$.

3. (a) Define the Lagrangian L and state Lagrange's equations of motion for a system with n degrees of freedom. State also what is meant by an ignorable coordinate and show that whenever there exists an ignorable coordinate a corresponding conservation law exists. Show, by direct calculation that,

$$\frac{d}{dt}\left[\dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L\right] = -\frac{\partial L}{\partial t}.$$

Hence sate the condition on L for the quantity $\dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L$ to be conserved.

(b) A pendulum bob of mass m, in a uniform gravitational field with strength g, is suspended by a light rod of length l beneath a mass M. The mass M is constrained to move horizontally along a smooth, fixed rail. The pendulum swings in the vertical plane containing the rail. Choose suitable generalised coordinates for the system and obtain the Lagrangian. Show that one of the generalised coordinates is ignorable and write down the corresponding conservation law. To what familiar conservation law does it correspond, and why is it expected that this law holds for this system? Find, giving reasons, another conservation law and state the familiar law to which it corresponds.

4. (a) The dynamics of a system are governed by a Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$. Define the generalised momenta p_i and the Hamiltonian H of the system and derive Hamilton's equations of motion.

   (b) A particle of mass m moving at relativistic speeds has Lagrangian

$$L = -\left(1 - \dot{q}^2\right)^{1/2} - V(q),$$

   where V is the potential. Find the corresponding Hamiltonian and write out Hamilton's equations of motion.

   Let $V(q) = -q$. For a particle starting from rest at the origin (i.e. $q = 0$), solve Hamilton's equations to show

$$(q+1)^2 - t^2 = 1.$$

5. A rigid body rotates freely with angular velocity $\boldsymbol{\omega}$. Write down, without proof, the relation between angular momentum of the rigid body and its angular velocity. What are principal axes? Explain briefly how they can be determined for a rigid body.

If principal axes are chosen for the rigid body with principal moments of inertia A, B and C, where $A < B < C$, obtain the Euler equations

$$A\dot{\omega}_1 + (C - B)\omega_2\omega_3 = 0,$$

$$B\dot{\omega}_2 + (A - C)\omega_1\omega_3 = 0,$$

$$C\dot{\omega}_3 + (B - A)\omega_1\omega_2 = 0.$$

Use the Euler equations to show

$$A\omega_1^2 + B\omega_2^2 + C\omega_3^2 = 2T,$$

$$A^2\omega_1^2 + B^2\omega_2^2 + C^2\omega_3^2 = L^2.$$

where T and L are constants.

For the special case $L^2 = 2BT$ it may be shown that

$$B^2\dot{\omega}_2^2 = \frac{(2T - B\omega_2^2)^2(C - B)(B - A)}{AC} \qquad (*),$$

(there is no need to derive equation (*)). Consider a rigid body rotating with angular speed $\sqrt{2T/B}$ purely about the principal axis corresponding to moment of inertia B. Use equation (*) to show that this state of motion is an equilibrium (steady) solution of the Euler equations.

You may assume that equation (*) can be rescaled into

$$\left(\frac{d\xi}{d\tau}\right)^2 = (1 - \xi^2)^2,$$

where $\tau = \alpha t$ ($\alpha > 0$ is a constant) and $\omega_2 = (2T/B)^{1/2}\xi$. Solve this differential equation to find the general solution for ξ. It is known that the above equilibrium solution is unstable. Use the general solution for ξ to show that if the body is disturbed from equilibrium, the angular velocity tends asymptotically (i.e. $t \to \infty$) to the negative of its original value.

[*Hint:* $d(\tanh^{-1}x)/dx = (1 - x^2)^{-1}$.]

6. A symmetric top moves about a fixed point P located on the symmetry axis, and is acted on by a force derivable from a potential $V(\theta)$. Here θ is the angle made by the symmetry axis to some fixed line through P. The Lagrangian $L(\psi, \phi, \theta)$ for the top with moments of inertia A and C is, in the usual notation,

$$L = \frac{1}{2}A\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}C\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - V(\theta).$$

Describe briefly, with the aid of a sketch, the sort of motions represented by $\dot{\theta}$, $\dot{\phi}$ and $\dot{\psi}$.

Show that

$$\dot{\psi} + \dot{\phi}\cos\theta = n,$$
$$A\dot{\phi}\sin^2\theta + Cn\cos\theta = h,$$
$$A\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + 2V = 2E - Cn^2,$$

where n, h and E are constants. Give a physical interpretation of each of these conservation laws.

If $V = \mu\cot^2\theta$, show that

$$A^2\dot{u}^2 = A(2E - Cn^2)(1 - u^2) - (h - Cnu)^2 - 2A\mu u^2,$$

where $u = \cos\theta$.

Suppose that the motion of the rigid body is such that the above equation becomes

$$\lambda^2\dot{u}^2 = \alpha^2 - u^2,$$

where $\lambda > 0$ and $0 < \alpha < 1$ are constants. State the range over which the angle of nutation changes. Given that initially $\theta = \cos^{-1}\alpha$, find $\theta(t)$ in terms of λ and α.