UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M233: Analytical Dynamics

COURSE CODE	:	**MATHM233**
UNIT VALUE	:	**0.50**
DATE	:	**21-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

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TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Let $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\hat{B} = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ be two sets of right-handed orthonormal vectors.

   (i) Define the transition matrix H from $\hat{B}$ to B. Given the relations

$$\mathbf{e}_i = H_{ij}\hat{\mathbf{e}}_j \quad \text{and} \quad \hat{\mathbf{e}}_i = H_{ji}\mathbf{e}_j,$$

   prove that H is orthogonal.

   (ii) Show that the matrix $\dot{H}H^T$, where $\dot{H}_{ij} = d(H_{ij})/dt$, is skew-symmetric. Explain briefly how the angular velocity $\boldsymbol{\omega}$ of B relative to $\hat{B}$ can be obtained from H.

   (b) A square with side length $2a$ is made of thin light wire and lies flat in a horizontal plane. The square rotates with constant angular speed ω about a vertical axis through the centre of the square. A bead of mass m slides without friction along one side of the square. Let x be the distance of the bead from the mid-point of the side of the square on which the bead is located. By analysing the motion within a frame of reference rotating with the square show that

$$\ddot{x} = \omega^2 x.$$

   At $t = 0$ the bead is located at the midpoint of one side of the square. It is observed that after one complete revolution of the square the bead just reaches an adjacent vertex. Show that the initial speed of the bead was $a\omega(\sinh 2\pi)^{-1}$.

2. (a) Consider a system of N particles of constant masses $m_1, m_2, \ldots, m_N$ and position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$. The external force acting on the ith particle is $\mathbf{F}_i^{(e)}$ and $\mathbf{F}_{ji}$ is the internal force acting on the ith particle due to the jth particle.

   (i) Define the centre of mass $\mathbf{R}$.

   (ii) Show that

$$M\ddot{\mathbf{R}} = \sum_{i=1}^{N} \mathbf{F}_i^{(e)},$$

   where M is the total mass of the system.

   (b) Two particles have masses m_1 and m_2, position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ and are acted upon by external forces $\mathbf{F}_1^{(e)}$ and $\mathbf{F}_2^{(e)}$. The internal force on particle 1 due to particle 2 is $\mathbf{F}_{21}$. If $\mathbf{F}_1^{(e)}/m_1 = \mathbf{F}_2^{(e)}/m_2$ show that

$$\mu\ddot{\mathbf{r}} = \mathbf{F}_{21},$$

   where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$ and $\mu = m_1 m_2/(m_1 + m_2)$.

   (c) Two masses m and M in a uniform gravitational field are joined by a massless spring with spring constant k and equilibrium length l. The two masses are aligned vertically and are released from rest with the spring compressed by an amount e from its equilibrium length. Find the spring extension as a function of time.

3. (a) Define the Lagrangian L and state Lagrange's equations of motion. State also what is meant by an ignorable coordinate and show that whenever there exists an ignorable coordinate a corresponding conservation law exists.

   (b) A particle of mass m moves on the surface of circular cone with vertex angle 2α. The base of the cone rests on a horizontal surface and the particle is attached to the vertex by a massless spring with spring constant k and equilibrium length l ($\ll$ the height of the cone). Let r be the distance of the particle from the symmetry axis of the cone. Show using Lagrange's equations that

$$\ddot{r} - \frac{h^2}{r^3} + \omega^2 r = \sin\alpha(g\cos\alpha + \omega^2 l),$$

   where g is the acceleration due to gravity, h is a constant and $\omega^2 = k/m$.

4. (a) The dynamics of a system are governed by a Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$. If the kinetic energy T is given by $T = \frac{1}{2}T_{ab}\dot{q}_a\dot{q}_b$, where $T_{ab} = T_{ab}(\mathbf{q})$, show that

$$\dot{q}_k\frac{\partial L}{\partial \dot{q}_k} - L$$

is equal to the total energy. To what symmetry does energy conservation correspond to?

(b) Define the generalised momenta p_i and the Hamiltonian H of the system. Write down Hamilton's equations of motion. Show that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}.$$

(c) A system with two degrees of freedom has Lagrangian

$$L = \frac{\dot{q}_1^2}{2q_2^2} + \frac{\dot{q}_2^2}{2} - \frac{q_2^2}{2}.$$

Find the corresponding Hamiltonian. Show that

$$q_2 = A\sin[(1+p^2)^{1/2}t + \epsilon],$$

where A, p and ϵ are constants.

5. (a) A rigid body has density ρ and volume V. Write down the expression for J_{ij}, the ijth element of the inertia matrix J calculated from a rest frame with origin O and set of axes $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Another set of axes $B' = \{\mathbf{e'}_1, \mathbf{e'}_2, \mathbf{e'}_3\}$ is chosen at O. Show that

$$J_{ij} = H_{ip} J'_{pq} H_{jq},$$

where H is the transition matrix from B' to B.

With reference to the above relation, explain why it is always possible to choose a coordinate system such that the inertia matrix is diagonal.

(b) Consider a body moving with zero applied torque. Write down, without proof, the relation between the angular momentum of the body and its angular velocity. If principal axes are chosen such that $J_{11} = A$, $J_{22} = B$ and $J_{33} = C$, obtain the Euler equation

$$A\dot{\omega}_1 + (C - B)\omega_2\omega_3 = 0.$$

Find also the Euler equations involving $\dot{\omega}_2$ and $\dot{\omega}_3$.

Suppose that the body with moments of inertia $A = B \neq C$ spins about the $\mathbf{e}_3$ principal axis such that $\omega_3 = \omega$ and $\omega_1 = \omega_2 = 0$. By substituting $\omega_1 = \xi_1$ and $\omega_2 = \xi_2$, where ξ_1 and ξ_2 are small, into Euler's equations show that the motion of the body is stable.

6. A symmetric top moves about a fixed point P in a uniform gravitational field. Explain what is meant by the angles of precession ϕ and nutation θ.

The Lagrangian $L(\psi, \phi, \theta)$ for a symmetric top with moments of inertia A and C is, in the usual notation,

$$L = \frac{1}{2}A\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}C\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - mga\cos\theta,$$

where a is the distance from the centre of mass to P.

What, physically, is $\dot{\psi}$ in this Lagrangian.

Derive three conservation laws for the motion of the top in the form of three equations. Give a physical interpretation for each of these equations.

A top is released with initial conditions $\theta = \pi/4$, $\dot{\theta} = \dot{\phi} = 0$ and $\dot{\psi} = \omega$. Show that, initially, the top begins to 'fall' i.e. θ increases. It is observed that the symmetry axis of the top reaches, but never falls below, the horizontal plane during its motion. Show that

$$C^2\omega^2 = 2\sqrt{2}mgaA.$$

Sketch the trajectory that the intersection of the symmetry axis of the top makes with the unit sphere in this case.