# UNIVERSITY COLLEGE LONDON

*University of London*

## EXAMINATION FOR INTERNAL STUDENTS

*For The Following Qualifications:-*

*B.Sc.* *M.Sci.*

**Mathematics M233: Analytical Dynamics**

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| COURSE CODE | : | **MATHM233** |
| UNIT VALUE | : | **0.50** |
| DATE | : | **12-MAY-03** |
| TIME | : | **10.00** |
| TIME ALLOWED | : | **2 Hours** |

03-C0953-3-50


**TURN OVER**

*All questions may be attempted but only marks obtained on the best* **four** *solutions will count.*
*The use of an electronic calculator is* **not** *permitted in this examination.*

1. (a) Let $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\hat{B} = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ be two sets of right-handed orthonormal vectors.

(i) Define the transition matrix $H$ from $\hat{B}$ to $B$. Given that $H$ is orthogonal, show that the matrix $\dot{H}H^T$, where $\dot{H}_{ij} = d(H_{ij})/dt$, is skew-symmetric.

(ii) Explain briefly how the angular velocity $\boldsymbol{\omega}$ of $B$ relative to $\hat{B}$ can be obtained from $H$.

(iii) Write down, without proof, the relationship between $D\mathbf{x}$ and $\hat{D}\mathbf{x}$, the time-derivatives of $\mathbf{x}$ with respect to $B$ and $\hat{B}$ respectively.

(b) In an inertial frame of reference the equation of motion of a particle of charge $q$ and mass $m$ orbiting about a fixed charge $-q'$ and subject to a uniform magnetic field $\mathbf{B}$ is

$$m\hat{D}^2\mathbf{r} = -\frac{k}{r^3}\mathbf{r} + q(\hat{D}\mathbf{r}) \times \mathbf{B},$$

where $\mathbf{r}$ is the position vector of charge $q$ from $-q'$, $r = |\mathbf{r}|$ and $k$ is a constant. Show that the equation of motion becomes

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{k}{mr^3}\mathbf{r} + \left(\frac{q}{2m}\right)^2 \mathbf{B} \times (\mathbf{B} \times \mathbf{r})$$

in a reference frame rotating uniformly with angular velocity $\boldsymbol{\omega}$ (to be found). For sufficiently weak $|\mathbf{B}|$ this equation reduces to

$$\frac{d^2\mathbf{r}}{dt^2} = -\frac{k}{mr^3}\mathbf{r}.$$

Describe the motion from the point of view of the laboratory (inertial) frame in this case.

[*Hint: this last equation describes an inverse square law with $q$ having an elliptical orbit about $q'$, where $q'$ is at one focus of the ellipse.*]

2. (a) Consider a system of $N$ particles of constant masses $m_1, m_2, \ldots, m_N$ and position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$. The external force acting on the $i$th particle is $\mathbf{F}_i^{(e)}$ and $\mathbf{F}_{ji}$ is the internal force acting on the $i$th particle due to the $j$th particle.

(i) Define the centre of mass $\mathbf{R}$.

(ii) Show that

$$M\ddot{\mathbf{R}} = \sum_{i=1}^{N} \mathbf{F}_i^{(e)},$$

where $M$ is the total mass of the system.

(iii) Show that the total kinetic energy can be written as $T = T_{CM} + T_{rel}$, where $T_{CM}$ is the kinetic energy of the centre of mass and $T_{rel}$ is kinetic energy about the centre of mass.

(b) Two particles of masses $m_1$ and $m_2$ move in the $(x, z)$-plane and are each subject to an external uniform gravitational field such that $\mathbf{F}_i^{(e)} = -m_i g\mathbf{k}$, $i = 1, 2$. In addition the particles feel a force of *attraction* of magnitude $\alpha/r^2$ where $\alpha$ is a constant and $r$ is the distance between the particles. If the system is released from *rest*, explain why only $r$ and the $z$-coordinate, $z_{CM}$, of the centre of mass are needed to describe the system. Show that

$$T_{rel} = \frac{1}{2}\mu\dot{r}^2,$$

where $\mu = m_1 m_2/(m_1 + m_2)$. Construct the Lagrangian of the system using $r$ and $z_{CM}$ as generalised coordinates. Use Lagrange's equations to verify that the relation in part (a)(ii) above holds for this system.

3. (a) Define the Lagrangian $L$. A system has kinetic energy which is purely a quadratic in the generalised velocities. Write down a constant of the motion for this system in the cases (i) $L$ is independent of one of the generalized coordinates and (ii) $L$ is independent of time $t$.

(b) A small bead of mass $m$ slides freely on a smooth uniform circular wire of radius $a$. The wire is *free* to rotate about a vertical diameter and has moment of inertia of $Ma^2/2$ about the rotation axis. Gravity acts in the downward vertical direction with acceleration $g$. The radius vector from the centre of the wire circle to the bead makes an angle $\theta$ with the downward vertical. Initially $\theta = \pi/2$ and $\dot{\theta} = 0$ and the angular velocity of the wire is $\dot{\phi} = \omega$.

How many degrees of freedom does the system have? Find the Lagrangian for the system and write down two conservation laws. Show that the angular velocity of the wire is

$$\dot{\phi} = \frac{M + 2m}{M + 2m\sin^2\theta}\omega.$$

Show also that the bead does not reach the bottom of the wire (i.e. $\theta = 0$) if

$$\omega^2 > \frac{2gM}{a(M + 2m)}.$$

4. (a) The dynamics of a system are governed by a Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$. Define the generalised momenta $p_i$ and the Hamiltonian $H$ of the system. Show that

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \dot{q}_i = \frac{\partial H}{\partial p_i}.$$

(b) A particle of mass $m$ moving in the $(x, y)$-plane with generalised momenta $p_x$ and $p_y$ has Hamiltonian

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + \omega(yp_x - xp_y),$$

where $\omega$ is a constant. Find the Lagrangian for the system and write out Lagrange's equations of motion. Comment on the state of motion of the observer for whom the above Hamiltonian is applicable.

5. (a) A rigid body with density $\rho$ and volume $V$ rotates freely with angular velocity $\boldsymbol{\omega}$ about a fixed point $P$. Show that it has angular momentum $\mathbf{L}_P$ about $P$ given by

$$L_{Pi} = J_{ij}\omega_j,$$

where

$$J_{ij} = \int_V \rho(r_k r_k \delta_{ij} - r_i r_j)dV,$$

is the $ij$th element of the inertia matrix. Explain why it is always possible to choose a coordinate system such that the inertia matrix is diagonal.

[*Hint*: $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$.]

(b) Consider a body moving with zero applied torque. If principal axes are chosen such that $J_{11} = A$, $J_{22} = B$ and $J_{33} = C$, obtain the Euler equation

$$A\dot{\omega}_1 + (C - B)\omega_2\omega_3 = 0.$$

Find also the Euler equations involving $\dot{\omega}_2$ and $\dot{\omega}_3$.

(c) A plane lamina, of possibly irregular shape and non-uniform density, moves freely in space about its centre of mass. Show that principal axes can be chosen such that $A + B = C$. Deduce that the component of angular velocity in the plane of the lamina has constant magnitude.

6. A symmetric top moves about a fixed point $P$ in a uniform gravitational field. Explain what is meant by nutation.

The Lagrangian $L(\psi, \phi, \theta)$ for a symmetric top with moments of inertia $A$ and $C$ is, in the usual notation,

$$L = \frac{1}{2}A\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}C\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - mga\cos\theta,$$

where $a$ is the distance from the centre of mass to $P$.

Find the three generalised momenta $p_\theta$, $p_\psi$ and $p_\phi$ and show $\dot{\phi} = (p_\phi - p_\psi \cos\theta)/(A\sin^2\theta)$. Show also that the Hamiltonian $H(p_\theta, p_\phi, p_\psi, \theta, \phi, \psi)$ is

$$H = \frac{1}{2A}p_\theta^2 + \frac{1}{2A\sin^2\theta}(p_\phi - p_\psi\cos\theta)^2 + \frac{1}{2C}p_\psi^2 + mga\cos\theta.$$

Use Hamilton's equations to deduce the existence of two conservation laws and give a physical interpretation for each of these laws. Write down a third conservation law.

Observe that the Hamiltonian can be written in the form

$$H = \frac{1}{2A}p_\theta^2 + U(\theta),$$

where $U(\theta)$ is an effective potential. Use Hamilton's equations to show that

$$A\ddot{\theta} = -\frac{dU}{d\theta}.$$

A top is put in motion with initially zero nutation and $\theta = \theta_0$, where $\theta_0$ is a (local) minimum of the effective potential $U(\theta)$. Using the above results find the motion of the top for all time.