UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics M233: Analytical Dynamics

COURSE CODE	:	**MATHM233**
UNIT VALUE	:	**0.50**
DATE	:	**14-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0948-3-50

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Let $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\hat{B} = \{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$ be two sets of right-handed orthonormal vectors. Define the transition matrix H from $\hat{B}$ to B. Given that H is orthogonal, show that the matrix $\dot{H}H^T$, where $\dot{H}_{ij} = d(H_{ij})/dt$, is skew-symmetric.
If
$$H = \begin{pmatrix} \cos(t^2) & 0 & -\sin(t^2) \\ 0 & 1 & 0 \\ \sin(t^2) & 0 & \cos(t^2) \end{pmatrix},$$
find the angular velocity of B relative to $\hat{B}$.

(b) The vectors in B and $\hat{B}$ are along the coordinate axes of two frames of reference R and $\hat{R}$ respectively. The two frames of reference share a common origin. The angular velocity of B relative to $\hat{B}$ is $\boldsymbol{\omega}$.
State, without proof, the Coriolis theorem relating $D\mathbf{x}$ and $\hat{D}\mathbf{x}$, the rates of changes of the vector $\mathbf{x}$ measured by observers fixed relative to R and $\hat{R}$ respectively.
A particle of mass m is subject to a force $\mathbf{F}$ in an inertial frame $\hat{R}$. Use the Coriolis theorem to derive a relation between the accelerations $D^2\mathbf{x}$ and $\hat{D}^2\mathbf{x}$ measured by observers fixed relative to R and $\hat{R}$ respectively. Hence write down the equation of motion of the particle in frame R. Identify the Coriolis and centrifugal force terms.

(c) Consider a Foucault's pendulum experiment taking place at the North pole of the Earth. Briefly describe and explain the behaviour of the pendulum according to, (i), an observer attached to the surface of the Earth and, (ii), an inertial observer.

2. (a) Consider a system of N particles of constant masses $m_1, m_2, \ldots, m_N$ and position vectors $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$. The external force acting on the ith particle is $\mathbf{F}_i^{(e)}$ and $\mathbf{F}_{ji}$ is the internal force acting on the ith particle due to the jth particle. What is the position vector of the centre of mass of the particles?

Write down the equation of motion for each particle and show that if there are no external forces acting on the particles then the centre of mass moves with constant velocity $\mathbf{v}$.

(b) The system of particles now moves under the influence of an external force. The internal forces $\mathbf{F}_{ji}$ are derivable from a potential $V_{\text{int}} = V_{\text{int}}[\mathbf{r}_1(t), \mathbf{r}_2(t), \ldots, \mathbf{r}_N(t)]$ i.e. $\sum_j \mathbf{F}_{ji} = -\nabla_i V_{\text{int}}$.

Show that the total kinetic energy T can be written as $T = T_{\text{CM}} + T_{\text{rel}}$, where T_{CM} is the kinetic energy of the centre of mass and T_{rel} is the kinetic energy about the centre of mass.

Show that

$$(i) \quad \frac{dT}{dt} = \sum_i \mathbf{v}_i \cdot \mathbf{F}_i^{(e)} + \sum_{i,j} \mathbf{v}_i \cdot \mathbf{F}_{ji},$$

$$(ii) \quad \frac{dT_{\text{CM}}}{dt} = \mathbf{v} \cdot \sum_i \mathbf{F}_i^{(e)},$$

$$(iii) \quad \frac{dV_{\text{int}}}{dt} = -\sum_{i,j} \mathbf{v}_i \cdot \mathbf{F}_{ji},$$

where $\mathbf{v}_i$ is the velocity of the ith particle.

If $\mathbf{F}_i^{(e)} = m_i \mathbf{c}$, $i = 1, 2, \ldots, N$, where $\mathbf{c}$ is a constant vector, deduce that $T_{\text{rel}} + V_{\text{int}}$ is a constant.

3. (a) Define the Lagrangian L for a system with n degrees of freedom. Write down Lagrange's equations of motion for the system.

   For a single particle moving in one dimension, deduce Newton's second law from Lagrange's equations.

   (b) A massless rod of length l is propped-up in a corner and is able to slide without friction. Its upper end is located at $(0, z)$ and its lower end at $(x, 0)$. A bead of mass m slides without friction on the rod under the influence of gravity acting in the $-\boldsymbol{k}$ direction. The motion of the bead and the rod is confined to the (x, z)-plane. Taking as generalized coordinates the distance s of the bead along the rod from its upper end and θ the angle the rod makes with the horizontal, show that the system has Lagrangian

$$L = \frac{m}{2}\left[\dot{s}^2 + s^2\dot{\theta}^2 - l\dot{s}\dot{\theta}\sin 2\theta + l(l-2s)\dot{\theta}^2\cos^2\theta\right] - mg(l-s)\sin\theta.$$

   Initially the rod is at rest (i.e. $\dot{\theta} = 0$), $\theta = \theta_0 \neq \pi/4$ and $s = l/2$. Use Lagrange's equations to show that intially

$$\ddot{\theta}(0) = -\frac{2g}{l}\frac{\cos\theta_0}{\cos 2\theta_0}.$$

   For what values of θ_0 does the upper end of the rod initially slide upwards?

4. (a) The dynamics of a system are governed by a Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$. Define the Hamiltonian H of a system and write down Hamilton's equations of motion. Show that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}.$$

(b) A particle of mass m moving in a plane is subject to a force $\mathbf{F} = F(r)\mathbf{e}_r$, where r is the distance of the particle from the origin and $\mathbf{e}_r$ is a unit vector in the radial direction.

Find the Hamiltonian of the system and use Hamilton's equations of motion to show that the angular momentum of the particle is conserved.

To what symmetry does this conservation law correspond?

(c) If a system has Hamiltonian

$$H = \frac{[p - F(q,t)]^2}{2G(q,t)} + V(q,t),$$

show that the corresponding Lagrangian is

$$L = \frac{1}{2}G(q,t)\dot{q}^2 + F(q,t)\dot{q} - V(q,t).$$

Consider the Hamiltonian

$$\bar{H} = \frac{\bar{p}^2}{2G(q,t)} + V(q,t) + \frac{\partial f(q,t)}{\partial t},$$

where $\bar{p} = G\dot{q}$ and $\partial f/\partial q = F$. Show that H and $\bar{H}$ give rise to the same motion.

5. (a) A rigid body with density ρ and volume V rotates freely with angular velocity $\boldsymbol{\omega}$ about a fixed point P.

Write down the expression for J_{ij}, the ijth element of the inertia matrix J calculated in a rest frame with origin P and set of axes $B = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$.

Show that the angular momentum $\mathbf{L}_P$ of the rigid body about P is given by

$$L_{Pi} = J_{ij}\omega_j.$$

Show also that the kinetic energy about P is

$$T = \frac{1}{2}\omega_i\omega_j J_{ij}.$$

Explain how, given J, principal axes at P can be found. What form does J take if principal axes are chosen as axes for the rest frame?

[You may assume the results

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c},$$

$$(\mathbf{a} \times \mathbf{b}) \cdot (\mathbf{a} \times \mathbf{b}) = (\mathbf{a} \cdot \mathbf{a})(\mathbf{b} \cdot \mathbf{b}) - (\mathbf{a} \cdot \mathbf{b})^2.]$$

(b) A rest frame has origin at the centre of a uniform rectangular block of mass m and is chosen such that the block is defined by $-a \le x \le a$, $-b \le y \le b$ and $-c \le z \le c$.

Show that the inertia matrix of the block in this rest frame is

$$J = \frac{m}{3}\begin{pmatrix} b^2 + c^2 & 0 & 0 \\ 0 & a^2 + c^2 & 0 \\ 0 & 0 & a^2 + b^2 \end{pmatrix}.$$

If the block spins with angular speed ω about an axis passing through the centre and a vertex of the block, calculate the block's kinetic energy about its centre of mass.

6. A symmetric top moves about a fixed point P in a uniform gravitational field.

The Lagrangian $L(\psi, \phi, \theta)$ for a symmetric top with moments of inertia A and C is, in the usual notation,

$$L = \frac{1}{2}A\left(\dot{\phi}^2 \sin^2\theta + \dot{\theta}^2\right) + \frac{1}{2}C\left(\dot{\psi} + \dot{\phi}\cos\theta\right)^2 - mga\cos\theta,$$

where a is the distance from the centre of mass to P.

Describe briefly, with the aid of a sketch, the sort of motions represented by $\dot{\theta}$, $\dot{\psi}$ and $\dot{\phi}$.

Show that

$$\dot{\psi} + \dot{\phi}\cos\theta = n,$$
$$A\dot{\phi}\sin^2\theta + Cn\cos\theta = h,$$
$$A\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + 2mga\cos\theta = 2E - Cn^2,$$

where n, h and E are constants. Give a physical interpretation of each of these conservation laws.

A top, initially spinning with angular velocity of magnitude ω about its symmetry axis, is released from rest with its symmetry axis nearly vertical (i.e. $\theta \approx 0$). The top is observed to move in such a way that the lowest position the symmetry axis reaches is horizontal. Use the above conservation laws to show that

$$\omega = \frac{1}{C}\sqrt{2mgaA}.$$

Sketch the trajectory that the intersection of the symmetry axis of the top makes with the unit sphere.