UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M211: Analysis 3: Complex Analysis

COURSE CODE	:	**MATHM211**
UNIT VALUE	:	**0.50**
DATE	:	**08-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1043-3-190

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. a) Find all complex numbers z such that $e^z = -1$.

   b) Show that the Cauchy-Riemann equations are satisfied for $f(z) = \frac{1}{z}$ on $\mathbb{C} \setminus \{0\}$.

   c) Is there a holomorphic function $F : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $F'(z) = \frac{1}{z}$? Justify your answer.

   d) Is there a holomorphic function $F : \mathbb{C} \setminus \{0\} \to \mathbb{C}$ such that $F'(z) = \frac{\cos z}{z^2}$? Justify your answer.

2. a) Define the terms *path* and *contour*.

   b) Define the path integral $\int_\gamma f(z)dz$.

   c) Compute $\int_\gamma \frac{1}{z}\, dz$ for $\gamma(t) = e^{it}, 0 \leqslant t \leqslant \pi$.

   d) Prove: If F is holomorphic on the open set G, $F'(z) = f(z)$ and $\gamma : [a, b] \to G$ is a path then $\int_\gamma f(z)dz = F(\gamma(b)) - F(\gamma(a))$.

   e) Is the following statement true or false? *If $f : G \to \mathbb{C}$ is holomorphic on an open set G and γ is a contour in G then* $\int_\gamma f(z)dz = 0$. Provide a short explanation.

3. a) State Cauchy's integral formula for a holomorphic function f and for its derivative f'.

   b) Derive Cauchy's estimate $|f'(z)| \leqslant \frac{1}{r} \max\{|f(\zeta)| : |\zeta - z| = r\}$.

   c) Derive Cauchy's integral formula for f from the residue theorem.

   d) Compute the integral $\int_\gamma \frac{e^z}{z(1+z)} dz$ where $\gamma(t) = \frac{1}{2}e^{it}, 0 \leqslant t \leqslant 2\pi$.

4. a) Define the following terms: *isolated singularity*, *pole*, *removable singularity*, *essential singularity*.

   b) Determine all singularities of $f(z) = \frac{\tan z}{z}$ and their type.

   c) Formulate Taylor's theorem for holomorphic functions.

   d) Find the Taylor series for $f(z) = \frac{1}{(z-i)^2}$ about $z = 0$ and determine its radius of convergence.

5. a) Define the residue $res(f, a)$. Formulate the residue theorem.

b) Show that the residue of $f(z) = \frac{p(z)}{q(z)}$ at a is given by $\frac{p(a)}{q'(a)}$ under suitable assumptions, and describe these assumptions.

c) Compute the integral $\int_{-\infty}^{\infty} \frac{1+x^2}{1+x^4}\, dx$.

6. Let $A = \{z : R < |z| < S\}$ be an annulus, and $f : A \to \mathbb{C}$ be a holomorphic function.

a) State Laurent's expansion theorem for f about 0.

b) Prove this theorem using Cauchy's integral formula for an annulus

$$f(z) = \frac{1}{2\pi i} \int_{\gamma_s} \frac{f(w)}{w - z} dw - \frac{1}{2\pi i} \int_{\gamma_r} \frac{f(w)}{w - z} dw, \qquad R < r < |z| < s < S$$

(where γ_c denotes the positively oriented circle with radius c and centre 0)

c) Find the Laurent expansion for $f(z) = \frac{z-1}{(z+1)(z-2)}$ about $z = 0$ in the annulus $\{z : 1 < |z| < 2\}$.

END OF PAPER