UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M211: Analysis 3: Complex Analysis

COURSE CODE : **MATHM211**

UNIT VALUE : **0.50**

DATE : **03-MAY-05**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

05-C0961-3-200

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let f be a differentiable function on the open disc $D(0,R)$ of centre 0 and radius R.

   Show

   (i) If $f'(z) = 0$ for all $z \in D(0,R)$, then f is constant on $D(0,R)$.
   (ii) If $|f|$ is constant on $D(0,R)$, then f is constant on $D(0,R)$.
   (iii) If $|f(z)| \leqslant |f(0)|$ for all $z \in D(0,R)$, then f is constant on $D(0,R)$.

2. Define

$$\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \quad \sin z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$
$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n!)}, \quad z \in \mathbb{C}.$$

   Show

   (i) $\exp z \neq 0, \quad \forall z \in \mathbb{C}$
   (ii) $\exp(a+b) = \exp a \exp b, \quad \forall a, b \in \mathbb{C}$
   (iii)

$$\cos z = \frac{1}{2}(e^{iz} + e^{-iz})$$
$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

   (iv) $\cos(a+b) = \cos a \cos b - \sin a \sin b, \quad \forall a, b \in \mathbb{C}.$
   (v) If $\sin z = 0$, then $z = k\pi$, where k is an integer.

3. State and prove Cauchy's Theorem for a triangle.

Let $\gamma : \{\gamma(\theta) = e^{i\theta}, 0 \leqslant \theta \leqslant 2\pi\}$ be the circle, centre 0 radius 1, taken in the anticlockwise direction. Evaluate $\int_\gamma f(z)dz$ when $f(z)$ is

(i) $\frac{1}{\sin z}$,

(ii) $\frac{1}{4z^2+1}$,

(iii) $4z^{-5}e^{-\frac{1}{z^4}}$

(iv) $\frac{1}{(e^z-1)^2}$

4. (i) State and prove Taylor's Theorem.

(ii) If f is holomorphic on $\mathbb{C}$ and f is not identically zero, show that the zeros of f are isolated, i.e. if $f(a) = 0$ then there exists $\delta > 0$ such that $f(z) \neq 0$ if $0 < |z - a| < \delta$.

(iii) Let f, g, h be holomorphic on $\mathbb{C}$ with $f\left(\frac{1}{2n}\right) = \frac{1}{4n^2}$, $g\left(\frac{1}{2n+1}\right) = \frac{-1}{(2n+1)^2}$, $h\left(\frac{1}{n}\right) = \frac{(-1)^n}{n^2}$, $n = 1, 2, \ldots$. Show that f, g exist but h does not.

5. (i) Find the Laurent expansion of $(z+1)^{-2}(z-1)^{-2}$ about the point -1 valid for

(a) $|z+1| < 2$,

(b) $|z+1| > 2$.

(ii) Given that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$, show that $\int_0^\infty \sin(x^2)dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

(iii) Evaluate $\int_0^\infty \frac{\cos x}{x^2+4}dx$.

6. (i) Let γ be a simple closed contour and f, g be holomorphic on and inside γ. If $|f(z)| > |g(z)|$, for all $z \in \gamma$, show that f and $f + g$ have the same number of zeros inside γ. Hence, or otherwise, deduce that any polynomial of degree n has n roots.

(ii) Evaluate

$$\sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)(5n+2)}.$$