UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M211: Analysis 3: Complex Analysis

COURSE CODE	:	**MATHM211**
UNIT VALUE	:	**0.50**
DATE	:	**05-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

04-C0955-3-190

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. (i) Let f be a complex valued function of $z = x + iy$, x, y real. Write $f(z) = u(x,y) + iv(x,y)$, $u(x,y), v(x,y)$ real. If f is differentiable at z, show that the Cauchy-Riemann equations $u_x = v_y$, $v_x = -u_y$ hold at z.

   (ii) Let $f(z) = |z|^2$. Show that f is differentiable only at 0.

   (iii) Let

$$g(z) = \frac{z^3}{|z|^2}, \quad z \neq 0.$$
$$g(0) = 0.$$

   Show that g is not differentiable at 0.

2. (i) Let f be differentiable on the open disc $D(0, R)$ of centre 0 and radius R. Show that

   a) if $f'(z) = 0 \ \forall z \in D(0, R)$, then f is constant,

   b) if $|f(z)|$ is constant on $D(0, R)$ then f is constant.

   (ii) If $\exp z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ show that

$$\exp(a + b) = \exp a . \exp b, \quad \forall a, b \in \mathbb{C}.$$

   (iii) If $\sin z = 1$, show that z is a real number.

3. (i) State and prove Taylor's theorem for a function f differentiable in the open disc $D(a, R)$, centre a, radius R.

   (ii) Find the Taylor expansion of $(1 - z)^{-1}$ of the form $\sum_{n=0}^{\infty} c_n (z + i)^n$, valid in the disc $D(-i, \sqrt{2})$.
   Would the expansion be valid in the disc $D(-i, 2)$?

   (iii) Find the Laurent expansion, about 0, of $(z^3 - 1)^{-1}$ valid a) for $|z| < 1$, b) $|z| > 1$.

4. (i) If f is holomorphic and bounded on $\mathbb{C}$, show that f is constant on $\mathbb{C}$.

(ii) Show that if $p(z)$ is a non-constant polynomial with coefficients in $\mathbb{C}$ then there exists $w \in \mathbb{C}$, with $p(w) = 0$.

(iii) If $q(z) = 7z^7 + 8z^6 + 3z^5 + 4z^4 + 1$, show that

$$\sup\{|q(z)|\,;|z| = 2\} > \sup\{|q(z)|\,;|z| = 1\}.$$

5. (i) Let f be a continuous function defined in the upper half plane $\operatorname{Im} z \geqslant 0$, where $f(z) \to 0$ uniformly as $|z| \to \infty$. If $m > 0$, show that

$$\int_{\Gamma} e^{imz} f(z)dz \to 0 \text{ as } R \to \infty$$

where $\Gamma_R = \left\{\mathrm{Re}^{i\theta},\ 0 \leqslant \theta \leqslant \pi\right\}$.

(ii) Evaluate

$$\int_0^\infty \frac{\sin x}{x(x^2+1)}dx.$$

6. (i) Let f be holomorphic inside and on a convex contour γ except for a finite number of poles at $a_1, ..., a_n$ inside γ with residues $R_1, ..., R_n$ respectively. Show that

$$\int_\gamma f(z)dz = 2\pi i(R_1 + ... + R_n).$$

(ii) Prove that

$$\int_0^{2\pi} \cos^{2n} x + \sin^{2n} x dx = \left(\frac{1}{2}\right)^{2n-2} \binom{2n}{n} \pi;$$

n a positive integer.