# UNIVERSITY COLLEGE LONDON

*University of London*

## EXAMINATION FOR INTERNAL STUDENTS

*For The Following Qualifications:-*

*B.Sc.* *M.Sci.*

**Mathematics M211: Analysis 3: Complex Analysis**

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| COURSE CODE | : | **MATHM211** |
| UNIT VALUE | : | **0.50** |
| DATE | : | **22-MAY-03** |
| TIME | : | **14.30** |
| TIME ALLOWED | : | **2 Hours** |

03-C0948-3-150

**All** *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. (i) Let $P(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0$ be a non-constant polynomial with complex coefficients $a_0, ..., a_n$, $n \geqslant 1$. Show that $P$ has at least one root in the complex plane. (Any results used must be clearly stated).

   (ii) If $a_0, ..., a_n$ are real and $P(w) = 0$, show that $P(\bar{w}) = 0$, where $\bar{w}$ is the complex conjugate of $w$. Is the same result necessarily true if $a_0, ..., a_n$ are complex numbers?

2. (i) Let

$$S = \{z : z = x + iy,\ xy > 1\}$$
$$T = \{z : z = x + iy,\ xy = 1\}.$$

   Show that $S$ is an open subset of $\mathbb{C}$ and $T$ is a closed subset of $\mathbb{C}$.

   (ii) Let $\{C_n\}_{n=1}^{\infty}$ be a nested sequence of non-empty sets in $\mathbb{C}$ i.e. $C_n \supset C_{n+1}$ for $n = 1, 2, ....$

   Decide, with justification, whether it is always true that $\bigcap_{n=1}^{\infty} C_n \neq \phi$ :

   a) When each $C_n$ is compact.

   b) When each $C_n$ is closed.

   c) When each $C_n$ is open.

3. (i) Let $f$ be a differentiable function on the complex plane $\mathbb{C}$ with $f'(z) = 0,\ \forall z \in \mathbb{C}$. Show that $f$ is constant on $\mathbb{C}$.

   (ii) Show that $\frac{d}{dz}(\exp z) = \exp z,\ \forall z \in \mathbb{C}$.

   (iii) Show that $\exp z \neq 0,\ \forall z \in \mathbb{C}$.

   (iv) Let $f$ be a diffentiable complex valued function on the real interval $[0, 1]$. Decide, with justification, if there always exists $c \in (0, 1)$ with

$$f(1) - f(0) = f'(c).$$

4. (i) State and prove Taylor's theorem.

   (ii) Find the Laurent expansion about 0 of the function $(4z^2 + 8z + 3)^{-1}$ valid

   a) for $\frac{1}{2} < |z| < \frac{3}{2}$.

   b) for $|z| > \frac{3}{2}$.

5. (i) State and prove Cauchy's Integral formula for a convex contour.

   (ii) Let $C$ denote the boundary of the square with vertices $\pm 3 \pm 3i$ taken in the anticlockwise direction. Evaluate

   a) $\displaystyle\int_C \frac{\sin z}{z(z^2+1)} dz.$

   b) $\displaystyle\int_C \frac{\cos z}{z^5} dz.$

6. (i) State and prove Jordan's lemma.

   (ii) Evaluate $\displaystyle\int_0^\infty \frac{x \sin x}{x^2+1} dx.$

   (iii) By integrating $\dfrac{\pi \operatorname{cosec} \pi z}{(2z+1)(2z-5)}$ around a suitable contour, evaluate

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(2n+1)(2n-5)}.$$



END OF PAPER