UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics M211: Analysis 3: Complex Analysis

COURSE CODE	:	**MATHM211**
UNIT VALUE	:	**0.50**
DATE	:	**30-APR-02**
TIME	:	**10.00**
TIME ALLOWED	:	**2 hours**

02-C0943-3-140

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. (i) Which of the following sequences converge?

   a) $\left\{\frac{1}{n^2}(1+i)^n\right\}_{n=1}^{\infty}$.

   b) $\{(-1)^n e^{in\pi}\}_{n=1}^{\infty}$.

   c) $\left\{\frac{z^n}{|z|^n}\right\}_{n=1}^{\infty}$, z not a real number.

   (ii) Define the concepts of open set and closed set in $\mathbb{R}^2$ and give an example of a set which is neither open nor closed in $\mathbb{R}^2$.

   (iii) Show that a bounded sequence of complex numbers has a convergent subsequence.

2. (i) Let $f = u + iv, u, v$ real, be a function which is differentiable at the point $z = x + iy$. Show that the Cauchy-Riemann equations $u_x = v_y$, $u_y = -v_x$ hold at z.

   (ii) If f is a differentiable function on $\mathbb{C}$ with $f(z)$ always purely imaginary, i.e. $f(z) = iv$, v real, show that f is constant on $\mathbb{C}$.

   (iii) If $f(z) = \frac{z^2}{|z|}$, $z \neq 0$, $f(0) = 0$, show that f is not differentiable at 0.

3. (i) Let f be a function continuous on and inside a triangle T and differentiable on and inside T except possibly at one point b. Show that

$$\int_T f(z)dz = 0.$$

   (ii) Let f be a function continuous on and inside the rectangle R with vertices $\pm 1 \pm i$, and differentiable at all points of R except possibly at the points $\left\{\frac{1}{n}(\pm 1 \pm i)\right\}_{n=1}^{\infty}$. Show that

$$\int_R f(z)dz = 0.$$

   (iii) Integrate the following functions around the unit circle, taken in the anticlockwise direction.

$$z^3,\ z^{-3},\ (z + z^{-1})^3,\ |z|^3.$$

4. (i) State and prove Laurent's theorem for a function f differentiable in the annulus $a < |z| < b$, a, b positive numbers, $a < b$.

(ii) Find the Laurent expansion about 0 of $(z^3 - 1)^{-1}$ valid for
(a) $|z| < 1$, (b) $|z| > 1$.

(iii) Find the principal part of the Laurent expansion about 0 of $\operatorname{cosec}^2(z)$.

5. (i) Let $f(z) = \dfrac{1}{(3z+2)(4z+1)}$. By integrating $(\pi \operatorname{cosec} \pi z)\, f(z)$ around a suitable square, evaluate

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n}{(3n+2)(4n+1)}.$$

(ii) Find the number of zeros of $z^4 + z^3 + z^2 + z + 2$ lying in the first quadrant i.e., $\{x + iy,\ x > 0,\ y > 0\}$.

6. (i) Evaluate

$$\int_0^\infty \frac{x \sin x}{1 + x^2} dx.$$

(ii) Evaluate the inverse Laplace transform of $\dfrac{1}{z^3+1}$, i.e. evaluate $F(t) = \dfrac{1}{2\pi i} \displaystyle\int_{1-i\infty}^{1+i\infty} \frac{e^{zt}}{z^3+1} dz$, where integration is along the line $(1 - i\infty,\ 1 + i\infty)$.