UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M11B: Analysis 2

COURSE CODE : **MATHM11B**

UNIT VALUE : **0.50**

DATE : **09-MAY-06**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

06-C1035-3-210

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) What does it mean to say that a function f at is differentiable at a?
   Suppose $f(x) = x^2 \sin \frac{1}{x}$ for $x \neq 0$, $f(0) = 0$.
   Show that f is differentiable at 0. Is f' continuous at 0? (Justify your answer.)

   (b) State the Chain Rule. Differentiate the following functions, with respect to x:
   (i) $\exp(x^2)$,
   (ii) $\cos(\sin(x^3))$,
   (iii) x^x.

2. Suppose f is continuous and strictly increasing on [a,b] . Let $y \in (f(a), f(b))$ and let $x = f^{-1}(y)$. Then, if f is differentiable at x and $f'(x) \neq 0$, prove that f^{-1} is differentiable at y and

$$(f^{-1})'(y) = \frac{1}{f'(x)}.$$

(You may assume that f^{-1} is continuous at y.)

If $g(x) = \sin^{-1}(\cos 2x)$, show that $g'(x)$ is equal to a constant for $x \in (0, \pi/2)$ and equal to another constant for $x \in (-\pi/2, 0)$.

Sketch the graph of $y = \sin^{-1}(\cos 2x)$, $-\pi/2 \leqslant x \leqslant \pi/2$.

3. (a) State Cauchy's Mean Value Theorem and use it to prove L'Hôpital's Rule.

   (b) Evaluate the following limits:
   (i) $\dfrac{1 - \cos x}{x^2}$ as $x \to 0$,
   (ii) $\dfrac{1}{x} - \dfrac{1}{\sin x}$ as $x \to 0$,
   (iii) $\dfrac{\log(1 + x)}{x}$ as $x \to 0$,
   (iv) $\left(1 + \dfrac{1}{x}\right)^x$ as $x \to \infty$,

4. State Rolle's Theorem. Prove the Mean Value Theorem. (You may assume Rolle's Theorem.)

   Suppose

   (a) $f(x) = x$ has a root α (that is, $f(\alpha) = \alpha$),
   (b) $f'(\alpha) = 0$,
   (c) $|f''(x)| \leqslant K$ for some K in a closed interval $[\alpha - h, \alpha + h]$, for some $h > 0$,

   Show that, if x_0 is sufficiently close to α, then $|\alpha - f(x_0)| < K|\alpha - x_0|^2$.

   Use Newton's Method to find a better approximation than 50 to $\sqrt{2501}$.

5. Suppose f is a bounded function on an interval $[a, b]$. Define the Upper Riemann Sum, $U(P, f)$, and the Lower Riemann Sum, $L(P, f)$, where P is a partition:

$$a = x_0 < x_1 < \cdots < x_n = b.$$

   What is meant by a refinement of P ? If P^* is a refinement of P show that $L(P, f) \leqslant L(P^*, f)$ and $U(P, f) \geqslant U(P^*, f)$.

   Hence show that if P and P' are partitions of $[a, b]$ then $L(P, f) \leqslant U(P', f)$.

   Suppose $P(n)$ are partitions of $[a, b]$, where $b > 0$, of the form $x_r = ah^r$, $r = 0, \ldots, n$, $ah^n = b$..

   Write down $U(P(n), x^2)$ and show directly (that is, without using the fact that x^2 is Riemann integrable) that as $n \to \infty$, $U(P(n), x^2) \to (b^3 - a^3)/3$.

6. State and prove the Integral Test for series. Find the limit of

$$\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{2n}$$

   as $n \to \infty$.

END OF PAPER