UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. B.Sc.(Econ)M.Sci.

Mathematics M11B: Analysis 2

COURSE CODE	:	**MATHM11B**
UNIT VALUE	:	**0.50**
DATE	:	**21-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0940-3-190

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. Let f, g be functions defined on (a, b), both of which are differentiable at a point $c\epsilon(a, b)$.

   (i) Show that $(fg)'(c) = f(c)g'(c) + f'(c)g(c)$.

   (ii) If also $g(c) \neq 0$, show that

$$(f/g)'(c) = \frac{f'(c)}{g(c)} - \frac{f(c)g'(c)}{(g(c))^2}.$$

   Let

$$h(\theta) = \frac{\sin\theta}{\theta}, \ \theta \neq 0 \quad . \qquad h(0) = 1$$

   Show that h is differentiable at 0. Is $h'(\theta)$ continuous at 0?

2. Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ be a power series with radius of convergence R.

   (i) Show that $\sum_{n=1}^{\infty} n\, a_n x^{n-1}$ also has radius of convergence R.

   (ii) Show that $f'(x) = \sum_{n=1}^{\infty} n\, a_n x^{n-1}, |x| < R$.

3. (i) State and prove Rolle's Theorem.

   (ii) If f is a differentiable function on (a, b) and $f'(x) > 0$, $\forall x\epsilon(a, b)$, show that f is strictly increasing on (a, b). If g is a differentiable function which is strictly increasing on (a, b), must $g'(x)$ be positive $\forall x\epsilon(a, b)$? Justify your answer!

   (iii) If f is a differentiable function on (a, b) and $m = \inf_{x\epsilon(a,b)} f'(x)$, $M = \sup_{x\epsilon(a,b)} f'(x)$ show that for any number λ, with $m < \lambda < M$, there exists $c\epsilon(a, b)$ with $f'(c) = \lambda$.

4. (i) Let f be a continuous function on $[a, b]$. Show that f is uniformly continuous on $[a, b]$, i.e. given $\varepsilon > 0, \exists \delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ if $x, y \epsilon [a, b]$ and $|x - y| < \delta$.

Deduce that f is Riemann integrable on $[a, b]$.

(ii) Let S_{10} denote the estimate for $\int_0^1 \sin(x^2)dx$ obtained by using Simpson's rule with $[0, 1]$ divided into 10 equal intervals. Show that

$$\left|\int_0^1 \sin(x^2)dx - S_{10}\right| < 10^{-4}.$$

5. Let f and g be Riemann integrable functions on $[a, b]$. Show that

(i) $f + g$; (ii) f^2; (iii) fg;

are Riemann integrable on $[a, b]$. (Any theorems used must be clearly stated).

Evaluate $\int_0^{\frac{\pi}{2}} x \cos^2 x \, dx$.

6. (i) Let $x_1, ..., x_n$ and $y_1, ...y_n$ be real numbers. Show that

$$\sum_{i=1}^{n} x_i y_i \leqslant \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} \left(\sum_{i=1}^{n} y_i^2\right)^{1/2},$$

(ii) If f is Riemann integrable on $[a, b]$ and $\int_a^b (f(x))^2 dx = 0$, show that $\int_a^b f(x)dx = 0$.

(iii) Show that if f and g are Riemann integrable on $[a, b]$, then

$$\int_a^b f(x)g(x)dx \leqslant \left(\int_a^b (f(x))^2 dx\right)^{1/2} \left(\int_a^b (g(x))^2 dx\right)^{1/2}.$$