UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. M.Sci.

Mathematics M11B: Analysis 2

COURSE CODE	:	**MATHM11B**
UNIT VALUE	:	**0.50**
DATE	:	**10-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0935-3-170

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.*

1. (i) Let f be defined on an open interval containing the point a. Define what it means for f to be differentiable at a.

   (ii) If

$$f(x) = x\cos\tfrac{1}{x}, \quad x \neq 0$$
$$f(0) = 0,$$

   show that f is continuous at 0 but not differentiable at 0.

   (iii) Let f, g be defined in an open interval containing a. If $h(x) = f(x)g(x)$ is differentiable at a, give an example in which f is differentiable at a but g is not.

2. (i) State and prove L'Hopital's rule.

   (ii) Evaluate

$$\lim_{x\to 0} \frac{x^5}{\sin^3 x - \tan^3 x}.$$

   (iii) If $f(x) = \sum_{n=1}^{\infty} x^n \cos\dfrac{\pi}{n}$, evaluate $\lim_{h\to 0} \dfrac{f(h) - f(-h)}{h}$.

3. (i) Prove Newton's method for iterative processes, i.e. Let g be a twice differentiable function on $[\alpha - h,\ \alpha + h]$ where $h > 0$ and $g(\alpha) = 0$. Suppose also that there exists M, $0 \leqslant M < 1$ with $\left|\dfrac{g''(x)g(x)}{(g'(x))^2}\right| \leqslant M$, $\forall x \in [\alpha - h,\ \alpha + h]$. Then, if $x_1 \in [\alpha - h,\ \alpha + h]$ and $x_{n+1} = x_n - \dfrac{g(x_n)}{g'(x_n)}$, $n = 1, 2, \cdots$, show that $x_n \to \alpha$ as $n \to \infty$.

   (ii) By choosing x_1 with $1.4 < x_1 < 1.5$, show that, if $g(x) = x^2 - 2$ and $x_{n+1} = x_n - \frac{(x_n^2 - 2)}{2x_n}$, then $x_n \to \sqrt{2}$ as $n \to \infty$.

4. Let f be a bounded function on $[a,b]$, $a < b$.

(i) Define the upper Riemann integral $\overline{\int_a^b} f(x)dx$ and the lower Riemann integral $\underline{\int_a^b} f(x)dx$ and show that

$$\underline{\int_a^b} f(x)dx \leqslant \overline{\int_a^b} f(x)dx.$$

(ii) Show that if f is continuous on $[a,b]$, $a < b$, then f is Riemann integrable on $[a,b]$. Find an example of a function g which is not continuous on $[a,b]$ but which, nevertheless, is Riemann integrable on $[a,b]$.

5. Let $\{G_\alpha\}_{\alpha \in A}$ be a collection of intervals whose union covers the interval I.

(i) If all the G_α are open intervals and I is a closed interval, show that there exists a finite subcollection of $\{G_\alpha\}_{\alpha \in A}$ whose union covers I.

(ii) If all the G_α are open intervals and I is an open interval, does there always exist a finite subcollection of $\{G_\alpha\}_{\alpha \in A}$ whose union covers I?

(iii) If all the G_α are closed intervals and I is a closed interval, does there always exist a finite subcollection of $\{G_\alpha\}_{\alpha \in A}$ whose union covers I?

6. (i) Let f be defined and continuous on $[a,b]$. If $\int_a^b f(x)dx$ exists and $x = g(t)$ with

(α) $a = g(c),\ b = g(d),\ g(t) \in [a,b]$ for $t \in [c,d]$,

(β) $g'(t)$ exists and is continuous in $[c,d]$.

Show that

$$\int_a^b f(x)dx = \int_c^d f(g(t))g'(t)dt.$$

(ii) Evaluate

$$\int_a^b \frac{x^2 dx}{\sqrt{1-x^3}},\quad 0 < a < b < 1.$$

(iii) Evaluate

$$\int_0^{\frac{\pi}{3}} \frac{\sin^3 x}{\sqrt{\cos x}} dx.$$

END OF PAPER