UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M11A: Analysis 1

COURSE CODE	:	**MATHM11A**
UNIT VALUE	:	**0.50**
DATE	:	**05-MAY-06**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

06-C1034-3-220

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. What does it mean to say that a sequence (x_n) converges to a limit L?

   Let (x_n) be an increasing sequence that is bounded above: prove that (x_n) converges.

   State and prove the Bolzano-Weierstrass Theorem. (You may assume an analogue of the second part of the question, for decreasing sequences.)

2. You are *given* the following fact: if $(x_k)_0^\infty$ is a sequence converging to s then

$$\frac{1}{n}\sum_{k=0}^{n-1} x_k \to s \quad \text{as } n \to \infty.$$

   Let the sequence (x_n) be defined inductively by

$$\begin{aligned} x_0 &= 0 \\ x_n &= 1 + \frac{1}{n}\sum_{k=0}^{n-1} x_k \quad \text{for } n \geqslant 1. \end{aligned}$$

   Prove that the sequence does *not* converge.

   Calculate the first few terms of the sequence. By considering the successive differences $x_n - x_{n-1}$, of the terms you have calculated, guess a formula for the general term x_n.

   Use the inductive definition to show that

$$nx_n - (n-1)x_{n-1} = x_{n-1} + 1$$

   and hence confirm your guess.

3. *Prove* that the series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges is $s > 1$.

For which real values of x does the following series converge?

$$\sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Justify your answer carefully.

[You may assume standard convergence tests provided that you state them clearly.]

4. For the purposes of this question you may assume that

$$\log x \leqslant x - 1 \quad \text{for every } x > 0. \tag{1}$$

Use inequality (1) to prove that

$$\log x \geqslant 1 - \frac{1}{x} \text{ for every } x > 0.$$

Show that for each natural number n

$$0 \leqslant \frac{1}{n} - \log\left(1 + \frac{1}{n}\right) \leqslant \frac{1}{n(n+1)}.$$

Deduce that the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \log\left(1 + \frac{1}{n}\right)\right)$$

converges and that the sum is at most 1.

Show that the sequence whose m^{th} term is

$$\left(\sum_{n=1}^{m} \frac{1}{n}\right) - \log(m+1)$$

converges to a number between 0 and 1.

5. *State and prove* the Cauchy-Schwarz inequality.

Let $(x_k)_1^n$ be a finite sequence of positive numbers. Show that

$$\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{x_k}\right)^{-1} \leqslant \frac{1}{n}\sum_{k=1}^{n} x_k. \qquad (2)$$

Assuming the AM/GM inequality,

$$\left(\prod_{k=1}^{n} a_k\right)^{1/n} \leqslant \frac{1}{n}\sum_{k=1}^{n} a_k$$

for positive numbers, show that the geometric mean

$$\left(\prod_{k=1}^{n} x_k\right)^{1/n}$$

lies between the two expressions in inequality (2).

6. What does it mean to say that a real function defined on **R** is continuous?

State and prove the intermediate value theorem for a continuous function $f : \mathbf{R} \to \mathbf{R}$.

Let $p : \mathbf{R} \to \mathbf{R}$ be the polynomial given by

$$p(x) = x^3 + x - 1.$$

Prove that there is a real number t with $p(t) = 0$.