UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *B.Sc.(Econ)M.Sci.*

Mathematics M11A: Analysis 1

COURSE CODE : **MATHM11A**

UNIT VALUE : **0.50**

DATE : **12-MAY-05**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

05-C0952-3-190
© 2005 University College London

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State and prove the Cauchy-Schwarz inequality.

   (b) Prove that for real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$,

$$\left(\sum_{i=1}^{n}(a_i + b_i)^2\right)^{1/2} \leqslant \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}.$$

   (c) Suppose that $a_1, \ldots, a_n$ are real numbers, and that $p_1, \ldots, p_n$ are nonnegative real numbers with $\sum_{i=1}^{n} p_i = 1$. Prove that

$$\sum_{i=1}^{n} p_i a_i \leqslant \left(\sum_{i=1}^{n} p_i a_i^2\right)^{1/2}.$$

2. (a) Define what it means for a sequence $(x_n)_{n=1}^{\infty}$ to be a *Cauchy sequence.*

   (b) Suppose that $|x_{n+1} - x_n| \to 0$ as $n \to \infty$. Must $(x_n)_{n=1}^{\infty}$ be a Cauchy sequence?

   (c) *State* the Bolzano-Weierstrass Theorem.

   (d) State and prove the General Principle of Convergence.

3. (a) Define what it means for a series $\sum_{n=1}^{\infty} x_n$ to be *convergent.*

   (b) *State* the Ratio Test for series.

   (c) State and prove the Comparison Test for series.

   (d) For each of the following series, determine whether or not it converges:

$$\sum_{n=1}^{\infty} \frac{n^{2005}}{2^n}, \qquad \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{3^n}, \qquad \sum_{n=3}^{\infty} \frac{2005n}{(n+1)^3 \log n}.$$

   [You may assume that $\sum_{n=1}^{\infty} 1/n^2$ converges.]

4. (a) Define $\exp(x)$ for a real number x by giving a power series.

(b) Define what it means for $\sum_{n=1}^{\infty} x_n$ to *converge absolutely.*

(c) Prove that the series for $\exp(x)$ converges absolutely.

(d) Prove that if $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ are absolutely convergent series then

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \left(\sum_{n=0}^{\infty} c_n\right),$$

where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$. [You may assume that any rearrangement of an absolutely convergent series converges to the same sum.]

(e) Prove that if x and y are real numbers then $\exp(x+y) = \exp(x)\exp(y)$. [You may use a general theorem about multiplication of series, provided that you state it carefully.]

5. (a) Define what it means to say that a function f is *continuous at c.*

(b) State and prove the Intermediate Value Theorem.

(c) Suppose that f and g are continuous real functions on $[0,1]$ such that

$$f(0) + f(1) = g(0) + g(1).$$

Prove that there is some $x \in [0,1]$ with $f(x) = g(x)$.

6. (a) Prove that a continuous function f on a closed interval $[a,b]$ is bounded.

(b) Prove that if f is continuous on $[a,b]$ and

$$M = \sup\{f(x) : x \in [a,b]\}$$

then there is $x \in [a,b]$ with $f(x) = M$.

(c) Suppose that f is continuous on $(0,1)$, and

$$M = \sup\{f(x) : x \in (0,1)\}.$$

Must there be $x \in (0,1)$ with $f(x) = M$?