UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. M.Sci.

Mathematics M11A: Analysis 1

COURSE CODE	:	**MATHM11A**
UNIT VALUE	:	**0.50**
DATE	:	**04-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

04-C0946-3-220

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define what it means for a sequence $(x_n)_{n=1}^{\infty}$ to converge to a limit a as $n \to \infty$.
   (b) Prove that if $x_n \to a$ and $y_n \to b$ as $n \to \infty$ then $x_n y_n \to ab$ as $n \to \infty$.
   (c) Prove that if $|r| < 1$ then $r^n \to 0$ as $n \to \infty$.
   (d) Determine the limit as $n \to \infty$ of

$$\frac{(2n+1)^{2004}(2n-1)^{2004}}{(2n^2+1)^{2004}}.$$

2. (a) Define what it means for a sequence $(x_n)_{n=1}^{\infty}$ to be a Cauchy sequence.
   (b) *State* the Bolzano-Weierstrass Theorem.
   (c) State and prove the General Principle of Convergence.

3. (a) Define what it means for a series $\sum_{n=1}^{\infty} x_n$ to be convergent.
   (b) State and prove the Comparison Test for series.
   (c) *State* the Ratio Test for series.
   (d) For each of the following series, determine whether or not it converges:

$$\sum_{n=1}^{\infty} \frac{2004^n}{n!}, \qquad \sum_{n=1}^{\infty} \frac{n^{2002}}{(n+1)^{2004}}, \qquad \sum_{n=1}^{\infty} \frac{(n+1)(2n+1)}{n^3}.$$

   [You may assume that $\sum_{n=1}^{\infty} 1/n^2$ converges and $\sum_{n=1}^{\infty} 1/n$ diverges.]

4. (a) Define $\exp(x)$ for a real number x by giving a power series.

(b) Define what it means for $\sum_{n=1}^{\infty} x_n$ to converge absolutely.

(c) Prove that the series for $\exp(x)$ converges absolutely.

(d) Prove that if x and y are real numbers then $\exp(x+y) = \exp(x)\exp(y)$. [You may use a general theorem about multiplication of series, provided that you state it carefully.]

(e) Prove that $1 + x \leqslant \exp(x)$ for all real x.

(f) Prove that if $\sum_{n=1}^{\infty} a_n$ is a convergent series of positive real numbers then the sequence $(b_n)_{n=1}^{\infty}$ defined by $b_n = \prod_{k=1}^{n}(1+a_k)$ is convergent. [Hint: Use (e) to show that $(b_n)_{n=1}^{\infty}$ is increasing and bounded above.]

5. (a) Define what it means to say that a function $f(x)$ is continuous at $x = c$.

(b) State and prove the Intermediate Value Theorem.

(c) Prove that the equation

$$\cos(x) = x\sin(x)$$

has infinitely many real solutions. [You may assume standard properties of cos and sin.]

6. (a) Define what it means for a function f to be convex on an interval I.

(b) State and prove Jensen's Inequality for convex functions.

(c) State and prove the Arithmetic Mean/Geometric Mean inequality. [You may assume that $-\log x$ is convex.]

(d) Prove that if $p_1, \ldots, p_n$ are positive real numbers with $\sum_{i=1}^{n} p_i = 1$ then

$$\sum_{i=1}^{n} p_i \log(1/p_i) \leqslant \log n.$$

END OF PAPER