UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc.   *M.Sci.*

Mathematics M11A: Analysis 1

COURSE CODE	:	**MATHM11A**
UNIT VALUE	:	**0.50**
DATE	:	**02-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0934-3-190

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) State and prove the Cauchy-Schwarz inequality.

   (b) Prove that, for real numbers $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$,

$$\left(\sum_{i=1}^{n}(a_i + b_i)^2\right)^{1/2} \leqslant \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}.$$

   (c) Prove that if $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ are positive real numbers then

$$\sum_{i=1}^{n} a_i b_i \leqslant \left(\sum_{i=1}^{n} a_i^{4/3}\right)^{3/4} \left(\sum_{i=1}^{n} b_i^4\right)^{1/4}.$$

   [Hint: Write $a_i b_i$ as $a_i^{2/3} \cdot (a_i^{1/3} b_i)$.]

2. (a) Define what it means for a sequence $(x_n)_{n=1}^{\infty}$ to converge to a limit a.

   (b) Define what it means for a to be an *upper bound* for a set S of real numbers. Define what it means for a to be the *least upper bound*.

   (c) *State* the Least Upper Bound Principle.

   (d) Prove that an increasing sequence $(x_n)_{n=1}^{\infty}$ that is bounded above converges to a limit.

   (e) State and prove the Bolzano-Weierstrass Theorem.

3. (a) Define what it means for a sequence $(x_n)_{n=1}^{\infty}$ to be a *Cauchy sequence*.

   (b) *State* the General Principle of Convergence.

   (c) State and prove the Comparison Test for series.

   (d) *State* the Ratio Test for series.

   (e) For each of the following series, determine whether or not it converges:

$$\sum_{n=1}^{\infty} \frac{3^n}{4^n n^2}, \qquad \sum_{n=1}^{\infty} \frac{5^n (n!)^2}{(2n)!}, \qquad \sum_{n=1}^{\infty} \frac{n!}{n^{n+2}}.$$

   [You may assume that $\sum_{n=1}^{\infty} 1/n^2$ converges.]

4. (a) Define $\exp(x)$ for a real number x by giving a power series.

(b) Define what it means for a series $\sum_{n=1}^{\infty} x_n$ to *converge*. Define what it means for $\sum_{n=1}^{\infty} x_n$ to *converge absolutely*.

(c) Prove that the series for $\exp(x)$ converges absolutely.

(d) Prove that if x and y are real numbers then $\exp(x+y) = \exp(x)\exp(y)$.

[You may use a general theorem about multiplication of series, provided that you state it carefully.]

(e) Prove that, for $x \geqslant 0$,

$$\exp(x) \geqslant 1 + x + \frac{x^2}{2}$$

and that, for $x \leqslant 0$,

$$\exp(x) \leqslant 1 + x + \frac{x^2}{2}.$$

[Hint: Consider the cases $x \leqslant -1$ and $-1 < x \leqslant 0$ separately. Note that $1+x+x^2/2$ is minimized at $x = -1$.]

5. (a) Define what it means to say that a function f is *continuous at* c.

(b) State and prove the Intermediate Value Theorem.

(c) Suppose f is a continuous real function such that $f(x) = f(x+2)$ for every real number x. Prove that there is a real number y such that $f(y) = f(y+1)$.

[Hint: Consider the function $g(x) = f(x) - f(x+1)$.]

6. (a) Prove that a continuous function f on a closed interval $[a, b]$ is bounded.

(b) Prove that if f is continuous on $[a, b]$ and

$$M = \sup\{f(x) : x \in [a, b]\}$$

then there is $x \in [a, b]$ with $f(x) = M$.

(c) Suppose that f and g are continuous functions on $[a, b]$ such that $f(x) > 0$ and $g(x) > 0$ for every $x \in [a, b]$. Prove that there is a constant c such that $f(x) \leqslant cg(x)$ for every $x \in [a, b]$.

END OF PAPER