UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C392: Algebraic Number Theory

COURSE CODE : **MATHC392**

UNIT VALUE : **0.50**

DATE : **05-MAY-06**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

06-C1023-3-50

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Let k be an algebraic number field. Define $N(x)$ and $\mathrm{Tr}(x)$ for $x \in k$.
   Prove that $N(x)$, $\mathrm{Tr}(x)$ are rational numbers.

   (b) Express the following symmetric polynomial $f(X_1, X_2, X_3)$ in terms of the elementary symmetric polynomials:

   $$f(X_1, X_2, X_3) = (X_1^2 + 1)(X_2^2 + 1)(X_3^2 + 1).$$

   Let $k = \mathbb{Q}(\alpha)$, where α has minimal polynomial $m(X) = X^3 + 2X^2 + 4X + 4$.
   Calculate $N(\alpha^2 + 1)$ and $N(\alpha^2 - 1)$.
   Determine whether $\alpha^2 + 1$ and $\alpha^2 - 1$ are irreducible in $\mathfrak{o}_k$.

2. (a) Prove that every algebraic number field has an integral basis.

   (b) Describe an algorithm for finding an integral basis.

   (c) Let $d \neq 1$ be a square-free integer and assume that $d \equiv 1 \bmod 4$.
   Show that $\frac{1+\sqrt{d}}{2}$ is an algebraic integer.
   Show that $\{1, \frac{1+\sqrt{d}}{2}\}$ is an integral basis in $\mathbb{Q}(\sqrt{d})$.

3. (a) Let $f \in \mathbb{Z}[X]$ be a monic polynomial of degree d, satisfying Eisenstein's criterion for the prime number p. For a zero α of f, define

   $$\theta = \frac{1}{p}\sum_{i=0}^{d-1} a_i\alpha^i, \qquad a_0, \ldots, a_{d-1} \in \{0, \ldots, p-1\}.$$

   Prove that if θ is an algebraic integer then $\theta = 0$.

   (b) Let p be a prime number and let $f(X) = X^p - p$. Let α be a zero of f.
   (i) Calculate $\left|\Delta\{1, \alpha, \ldots, \alpha^{p-1}\}\right|$ in terms of p.
   (ii) Hence show that $\{1, \alpha, \ldots, \alpha^{p-1}\}$ is an integral basis in $\mathbb{Q}(\alpha)$.

4. (a) Let $\mathfrak{o}$ be the ring of algebraic integers in an algebraic number field k.

   Define the norm $N(I)$ of a non-zero ideal I of $\mathfrak{o}$.

   Prove that $N(IJ) = N(I)N(J)$ for any two non-zero ideals $I, J \subseteq \mathfrak{o}$.

   (b) Let $k = \mathbb{Q}(\alpha)$, where α has minimal polynomial $m(X) = X^3 + 4X + 2$.

   Show that $\mathfrak{o}_k = \mathbb{Z}[\alpha]$.

   Factorize (2), (3) and (5) into maximal ideals of $\mathfrak{o}_k$.

   Find the norm of each of the maximal ideals and show that they are all principal ideals.

5. Calculate the class group of the field $\mathbb{Q}(\sqrt{-33})$, giving a representative ideal for each ideal class and a multiplication table for the group.

END OF PAPER