UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C392: Algebraic Number Theory

COURSE CODE	:	**MATHC392**
UNIT VALUE	:	**0.50**
DATE	:	**20-MAY-05**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

05-C0940-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define the term *algebraic integer.*

   Let $n \neq 0, 1$ be a square-free integer. Show that $\frac{1+\sqrt{n}}{2}$ is an algebraic integer if and only if $n \equiv 1 \bmod 4$.

   (b) Describe an algorithm for finding an integral basis in an algebraic number field.

   (c) Find an integral basis in the field $\mathbb{Q}(\alpha)$, where α has minimal polynomial $X^3 - 3X - 3$.

2. Let $p \geq 3$ be a prime number; let ζ be a primitive p-th root of unity; let $\lambda = \zeta - 1$, and let $k = \mathbb{Q}(\zeta)$.

   (a) Find the minimal polynomial of λ and show that it is irreducible.

   (b) Calculate $N(\zeta)$, $N(\lambda)$ and $|\Delta\{1, \lambda, \ldots, \lambda^{p-2}\}|$.

   (c) Show that the ring of algebraic integers in k is $\mathbb{Z}[\zeta]$.

3. (a) Let $\mathfrak{O}$ be the ring of algebraic integers in an algebraic number field k.

   Define the norm $N(I)$ of a non-zero ideal I of $\mathfrak{O}$.

   Show that for a non-zero principal ideal $I = (\alpha)$, we have

   $$N(I) = |N(\alpha)|.$$

   (you may assume if you wish that k is a splitting field over $\mathbb{Q}$).

   (b) Let I be a non-zero, principal ideal of $\mathbb{Z}[\sqrt{30}]$. Show that $N(I)$ is congruent to 0 or ± 1 modulo 5.

   Hence show that $\mathbb{Z}[\sqrt{30}]$ is not a principal ideal domain.

4. (a) State and prove Dedekind's prime factorization theorem.

   (b) Show that $\mathbb{Z}[\sqrt[3]{2}]$ is a principal ideal domain. You may assume that $\{1, \sqrt[3]{2}, \sqrt[3]{4}\}$ is an integral basis in $\mathbb{Q}(\sqrt[3]{2})$.

5. Calculate the class group of the field $\mathbb{Q}(\sqrt{-30})$, giving a representative ideal for each ideal class and a multiplication table for the group.