UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C392: Algebraic Number Theory

COURSE CODE	:	**MATHC392**
UNIT VALUE	:	**0.50**
DATE	:	**06-MAY-04**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

04-C0935-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) For an algebraic number field k define the terms:
   (i) a *field embedding*;
   (ii) the *norm of an element*;
   (iii) the *trace of an element*;
   (iv) the *discriminant of a basis.*

   (b) Describe an algorithm for finding an integral basis in k.

   (c) Let $d \in \mathbb{Z}$ be square-free and assume $d \neq 1$. Prove the following:
   (i) if $d \equiv 1 \bmod 4$ show that $\{1, \frac{1+\sqrt{d}}{2}\}$ is an integral basis in $\mathbb{Q}(\sqrt{d})$.
   (ii) if $d \not\equiv 1 \bmod 4$ show that $\{1, \sqrt{d}\}$ is an integral basis in $\mathbb{Q}(\sqrt{d})$.

2. (a) Let $f \in \mathbb{Z}[X]$ be a polynomial of degree d satisfying Eisenstein's criterion for the prime number p. For a zero α of f, consider the element:

$$\theta = \frac{1}{p}\sum_{i=0}^{d-1} a_i\alpha^i, \qquad a_0, \ldots, a_{d-1} \in \{0, \ldots, p-1\}.$$

   Prove that if θ is an algebraic integer and each $a_i \in \{0, 1, \ldots, p-1\}$ then $\theta = 0$.

   (b) Let p be a prime number and let $f(X) = X^4 - p$. Let α be a zero of f.
   (i) Calculate $|\Delta\{1, \alpha, \alpha^2, \alpha^3\}|$ in terms of p.
   (ii) By making the change of variable $g(X) = f(X+1)$ or otherwise, show that if $p \equiv 3 \bmod 4$ then $\{1, \alpha, \alpha^2, \alpha^3\}$ is an integral basis in $\mathbb{Q}(\alpha)$.

3. (a) Let $\mathfrak{O}$ be the ring of algebraic integers in an algebraic number field k.
Define the norm $N(I)$ of a non-zero ideal I of $\mathfrak{O}$.
Prove that $N(IJ) = N(I)N(J)$ for any two non-zero ideals $I, J \subseteq \mathfrak{O}$.
(b) Consider the ideal $I = \langle 3 + 3\sqrt{15} \rangle$ in the ring $\mathbb{Z}[\sqrt{15}]$.
(i) Calculate $N(I)$.
(ii) Factorize I into maximal ideals.
(iii) Determine which of the maximal ideals dividing I are principal ideals.

4. (a) State and prove Dedekind's prime factorization theorem.
(b) Let α be a zero of the polynomial $f(X) = X^3 + 2X + 2$.
(i) Show that $\mathbb{Z}[\alpha]$ is the ring of algebraic integers in $\mathbb{Q}(\alpha)$.
(ii) Factorize $\langle 2 \rangle$, $\langle 3 \rangle$ and $\langle 5 \rangle$ into maximal ideals in $\mathbb{Z}[\alpha]$.

5. Calculate the class group of the field $\mathbb{Q}(\sqrt{-21})$, giving a representative ideal for each ideal class and a multiplication table for the group.

END OF PAPER