UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C392: Algebraic Number Theory

COURSE CODE	:	**MATHC392**
UNIT VALUE	:	**0.50**
DATE	:	**01-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

03-C0929-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Define the term *integral basis* for an algebraic number field k.

   Describe an algorithm for finding an integral basis.

   (b) Let $d \in \mathbb{Z}$ be square-free and assume $d \neq 1$.

   Using your algorithm show that

   (i) if $d \equiv 1 \bmod 4$ then $\{1, \frac{1+\sqrt{d}}{2}\}$ is an integral basis in $\mathbb{Q}(\sqrt{d})$.

   (ii) if $d \not\equiv 1 \bmod 4$ then $\{1, \sqrt{d}\}$ is an integral basis in $\mathbb{Q}(\sqrt{d})$.

2. (a) Let $f \in \mathbb{Z}[X]$ be a polynomial of degree d satisfying Eisenstein's criterion for the prime number p. For a zero α of f, consider the element:

$$\theta = \frac{1}{p}\sum_{i=0}^{d-1} a_i\alpha^i, \qquad a_0, \ldots, a_{d-1} \in \{0, \ldots, p-1\}.$$

   Prove that if θ is an algebraic integer then $a_0 = \ldots = a_{d-1} = 0$.

   (b) Let $f(X) = X^d - a$ be irreducible and let α be a zero of f. Show that

$$\Delta\{1, \alpha, \ldots, \alpha^{d-1}\} = \pm(da)^{d-1}.$$

   (You may assume that $\Delta\{1, \alpha, \ldots, \alpha^{d-1}\} = \pm N(f'(\alpha))$).

   (c) For each of the following polynomials f, find an integral basis in $\mathbb{Q}(\alpha)$, where α is a zero of f.

   (i) $f(x) = x^3 - 4x - 2$;

   (ii) $f(x) = x^4 - 2$;

   (iii) $f(x) = x^3 - 2$.

3. (a) Let $\mathfrak{O}$ be the ring of algebraic integers in an algebraic number field k.

   Define the norm of an ideal of $\mathfrak{O}$.

   Prove that $N(IJ) = N(I)N(J)$ for any two ideals $I, J \subseteq \mathfrak{O}$.

   (b) Consider the ideal $I = \langle 2 + \sqrt{22} \rangle$ in the ring $\mathbb{Z}[\sqrt{22}]$.

   (i) Calculate $N(I)$.

   (ii) Factorize I into maximal ideals.

   (iii) Determine which of the factors of I are principal ideals.

4. (a) State and prove Dedekind's prime factorization theorem.

   (b) Factorize $\langle 2 \rangle$, $\langle 3 \rangle$ and $\langle 5 \rangle$ into maximal ideals in $\mathbb{Z}[\alpha]$, where α is a zero of the polynomial $X^3 - 3X + 3$.

5. Calculate the class groups of the field $\mathbb{Q}(\sqrt{-30})$, giving a representative ideal for each ideal class and a multiplication table for the group.