UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics C392: Algebraic Number Theory

COURSE CODE	:	**MATHC392**
UNIT VALUE	:	**0.50**
DATE	:	**29-APR-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0924-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Let $d \in \mathbb{Z}$ be square-free integer and suppose $d \neq 1$.

   Prove:

   (i) if $d \not\equiv 1 \bmod 4$ then $\{1, \sqrt{d}\}$ is an integral basis in $\mathbb{Q}(\sqrt{d})$;

   (ii) if $d \equiv 1 \bmod 4$ then $\{1, \frac{1+\sqrt{d}}{2}\}$ is an integral basis in $\mathbb{Q}(\sqrt{d})$.

   (b) Find an integral basis in the field $\mathbb{Q}(\alpha)$, where α is a zero of the polynomial $X^3 + 2X + 2$.

2. Let p be an odd prime number and let ζ be a primitive p-th root of unity. Let $\lambda = \zeta - 1$.

   (a) Write down the minimal polynomial of ζ.

   Hence find the minimal polynomial of λ.

   Find $N(\zeta)$ and $N(\lambda)$.

   (b) Consider the basis $\{1, \lambda, \ldots, \lambda^{p-2}\}$ of the field $\mathbb{Q}(\zeta)$.

   Show that $\Delta\{1, \lambda, \ldots, \lambda^{p-2}\} = \pm p^{p-2}$. Any formulae for discriminants which you use should be clearly stated.

   Prove that $\{1, \lambda, \ldots, \lambda^{p-2}\}$ is an integral basis in $\mathbb{Q}(\zeta)$. You may assume that λ^{p-1} divides p in the ring of algebraic integers in $\mathbb{Q}(\zeta)$.

3. (a) Let $\mathfrak{O}$ be the ring of algebraic integers in an algebraic number field k. Prove that for non-zero ideals I, J of $\mathfrak{O}$,

$$N(IJ) = N(I)N(J).$$

   (b) Calculate the norm of the ideal $I = \langle 6, 2 + \sqrt{-26} \rangle$ of $\mathbb{Z}[\sqrt{-26}]$.

   Hence or otherwise show that $\mathbb{Z}[\sqrt{-26}]$ is <u>not</u> a principal ideal domain.

   Factorize the ideal $\langle 6, 2 + \sqrt{-26} \rangle$ into prime ideals of $\mathbb{Z}[\sqrt{-26}]$.

4. (a) State and prove Dedekind's Prime Factorization Theorem.

   (b) Consider the field $k = \mathbb{Q}(\alpha)$, where α is a zero of the polynomial $f(X) = X^3 + X + 3$.

   Show that the ring of algebraic integers in k is $\mathbb{Z}[\alpha]$.

   Factorize the ideals $\langle 2 \rangle$, $\langle 3 \rangle$ and $\langle \alpha + 4 \rangle$ into maximal ideals of $\mathbb{Z}[\alpha]$.

5. Calculate the class group of the field $\mathbb{Q}(\sqrt{-30})$, giving a representative for each ideal class and a multiplication table for the group.

END OF PAPER