UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics M120: Algebra for Joint Honours Students

COURSE CODE	:	**MATHM120**
UNIT VALUE	:	**0.50**
DATE	:	**24-MAY-05**
TIME	:	**10.00**
TIME ALLOWED	:	**2 Hours**

05-C0954-3-50

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 1 & 3 & 5 & 6 & 2 \end{pmatrix}, \qquad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 5 & 6 & 1 \end{pmatrix}.$$

   (a) Express σ and τ in cycle notation.

   (b) Find (in standard notation) $\tau\sigma$, $\sigma\tau$, and σ^{-1}.

   (c) What are the orders of τ, σ, and $\tau\sigma$?

   (d) Give, with explanations, the signs of τ, σ, and $\tau\sigma$.

   (e) Express τ explicitly as a product of flips (transpositions). Draw a braid diagram for τ.

2. (a) Find the following numbers using modular arithmetic:

   (i) $23 \mod 3$;

   (ii) $7^{3285} \mod 6$;

   (iii) $5^{3285} \mod 6$;

   (iv) $9187236454 \mod 9$;

   (v) $17^{42} \mod 43$.

   (b) For each of the following functions, determine which functions are one-to-one (injective), and which functions are onto (surjective). If a function is not one-to-one or not onto, explain why not. The set $(-\pi/2, \pi/2)$ is the real interval $-\pi/2 < x < \pi/2$.

   (i) $f : \mathbb{R} \to \mathbb{R}, \qquad f(x) = x^3 - x$;

   (ii) $g : \mathbb{Z} \to \mathbb{R}, \qquad g(x) = x$;

   (iii) $h : (-\pi/2, \pi/2) \to \mathbb{R}, \qquad h(x) = \tan(x)$.

   (c) The symmetry group of a pentagon (the dihedral group D_5) has elements

$$\{e, r, r^2, r^3, r^4, f, rf, r^2f, r^3f, r^4f\},$$

   where r gives a rotation through $72°$ and f gives a flip.
   Which group elements listed above correspond to the following products:

   (i) fr^4;

   (ii) $(r^2f)^{-1}$;

   (iii) $r^5f^3r^{21}f^4rfrf$.

   (d) Find subgroups of D_5 of order 2 and 5.

3. (a) Let a, b, and c be elements of the group $\langle G, *\rangle$. Show that if $a * b = a * c$ then $b = c$.

   (b) Suppose H is a subgroup of G. Consider elements g and k in G, and let

$$gH = \{g * h \mid h \in H\}, \qquad kH = \{k * h \mid h \in H\}$$

   be left cosets of H. Suppose that g is not an element of kH. Show that gH has no elements in common with kH, i.e. the intersection of gH and kH is null.

   (c) Show that $|gH| = |kH| = |H|$.

   (d) Hence state and prove Lagrange's theorem.

4. (a) Describe three types of elementary row operations.

   (b) What is the augmented matrix corresponding to the system of linear equations

$$\begin{aligned} x - y + z &= -4; \\ 2y + y + 5z &= 1; \\ 3x - 2y - 2z &= 3. \end{aligned}$$

   (c) Transform the augmented matrix to reduced row echelon form. What is the solution (if any) to the system of equations?

5. (a) Consider a set of n vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$. What condition must these vectors satisfy in order for them to be linearly dependent?

   (b) Are the following three vectors linearly independent?

$$\begin{aligned} \mathbf{v}_1 &= (1,2,3); && (1) \\ \mathbf{v}_2 &= (2,3,4); && (2) \\ \mathbf{v}_3 &= (0,1,2). && (3) \end{aligned}$$

   (c) Find a basis for the vector space of 2×3 matrices.

   (d) Let $GL(n)$ be the group of invertible $n \times n$ matrices with matrix multiplication. Also let $O(n)$ be the set of orthogonal $n \times n$ matrices, i.e. matrices which satisfy $A^{-1} = A^T$. Show that $O(n)$ is a subgroup of $GL(n)$.

   (e) Suppose $A \in O(n)$ is an orthogonal matrix. Show that $|\det A| = 1$.

6. Let

$$A = \begin{bmatrix} 5 & 3 \\ -1 & 1 \end{bmatrix}$$

(a) Find all eigenvalues for A.

(b) Find the associated eigenvectors.

(c) Find an invertible matrix B and a diagonal matrix Δ such that $B^{-1}AB = \Delta$.

(d) Hence find a square root matrix P which satisfies the equation

$$P^2 = A.$$