University of London

## EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-
B.Sc. M.Sci.

Mathematics M120: Algebra for Joint Honours Students

COURSE CODE : MATHM120

UNIT VALUE : 0.50

DATE : 28-MAY-04

TIME : 10.00

TIME ALLOWED
: 2 Hours

All questions may be attempted but only marks obtained on the best four solutions will count.
The use of an electronic calculator is not permitted in this examination.

1. Let

$$
\rho=\left(\begin{array}{lllllll}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
2 & 5 & 4 & 6 & 7 & 3 & 1
\end{array}\right), \quad \sigma=\left(\begin{array}{lllllll}
1 & 2 & 3 & 4 & 5 & 6 & 7 \\
5 & 2 & 1 & 7 & 3 & 4 & 6
\end{array}\right) .
$$

(a) Express $\rho$ and $\sigma$ in cycle notation.
(b) Find (in standard notation) $\rho \sigma, \sigma \rho$, and $\rho^{-1}$.
(c) What are the orders of $\rho$ and $\sigma$ ?
(d) Give, with explanations, the signs of $\rho$ and $\sigma$.
(e) Let $A_{7}$ be the set of even permutations of seven elements. Show that $A_{7}$ is a subgroup of $S_{7}$, the group of all permutations of seven elements. What is the order of $A_{7}$ ?
2. (a) Let $\left\langle\mathbb{Z}_{11}^{*}, \bullet\right\rangle$ be the multiplicative group $\bmod 11($ where $a \bullet b \equiv a b \bmod 11)$. In this group, find the group elements corresponding to
(i) $5 \bullet 7$;
(ii) $3^{-1}$;
(iii) $10^{-1}$;
(iv) $10^{7}$.
(b) What is the order of the symmetry group of an $n$ sided regular polygon, $D_{n}$ ? What is the order of the symmetry group of a cube? What is the order of the symmetry group of a tetrahedron?
(c) The symmetry group of a triangle (the dihedral group $D_{3}$ ) has elements $\left\{e, r, r^{2}, f, r f, r^{2} f\right\}$, where $r$ gives a rotation through $120^{\circ}$ and $f$ gives a flip. Which group element corresponds to the following products:
(i) $f r$;
(ii) $(r f r)^{-1}$;
(iii) $r^{6} f^{3} r^{19} f^{4} r f r f r f$.
(d) Find subgroups of $D_{3}$ of order 2 and 3.
(e) Given two groups $\langle G, *\rangle$ with operation $*$ and $\langle H, \otimes\rangle$ with operation $\otimes$, what is the definition of a homomorphism from $G$ to $H$ ? What is an isomorphism from $G$ to $H$ ?
(f) Write down all elements of the permutation group on 3 objects, i.e. the symmetric group $S_{3}$ (you may use either standard notation or cycle notation). Find an isomorphism from $D_{3}$ to $S_{3}$.
3. (a) Given a system of linear equations whose corresponding augmented matrix has reduced row echelon form $M$, describe the conditions on $M$ which imply that the system has
(i) a unique solution,
(ii) no solution,
(iii) many solutions.
(b) What is the augmented matrix corresponding to the system of linear equations

$$
\begin{aligned}
2 x-3 y+2 z & =-1 \\
x-y-2 z & =4 \\
-x+2 y & =2
\end{aligned}
$$

Transform the augmented matrix to reduced row echelon form. What is the solution (if any) to the system of equations?
4. (a) Let $A$ and $B$ be square ( $n \times n$ ) matrices. Define, in terms of $i, j$ entries the product $A B$, the transpose $A^{T}$, and trace $(A)$.
(b) show that for three square matrices $P, Q$, and $R$,

$$
\operatorname{trace}(P Q R)=\operatorname{trace}(R P Q)
$$

Also show that

$$
\operatorname{trace}\left(B^{-1} A B\right)=\operatorname{trace}(A)
$$

for square matrices $A$ and $B$.
(c) Show that $(P Q)^{T}=Q^{T} P^{T}$.
(d) Show that matrix multiplication of square matrices is associative.
(e) Find the inverse of the matrix

$$
\left(\begin{array}{ll}
1 & 2 \\
3 & 4
\end{array}\right) .
$$

5. Let

$$
A=\left[\begin{array}{ll}
3 & 2 \\
2 & 0
\end{array}\right]
$$

(a) Find all eigenvalues for $A$.
(b) Find the associated eigenvectors.
(c) Find, if possible, an invertible matrix $B$ and a diagonal matrix $\Delta$ such that $B^{-1} A B=\Delta$.
(d) Let $M$ be a symmetric $n \times n$ matrix, $M=M^{T}$. Let V and W be $n$-dimensional vectors. Show that

$$
(M \mathbf{V}) \cdot \mathbf{W}=\mathbf{V} \cdot(M \mathbf{W})
$$

(e) Suppose $M$ is a symmetric $n \times n$ matrix, with distinct eigenvalues $a$ and $b$, corresponding to eigenvectors $\mathbf{V}$ and $\mathbf{W}$. Show that $\mathbf{V}$ and $\mathbf{W}$ are orthogonal.
6. Consider a square region $R$ in the $x-y$ plane, aligned parallel to the $x$ and $y$ axes. Let $\mathbf{V}$ be a two-dimensional vector representing a point in $R$.
(a) Consider the following linear transformations acting on $R$. Tell whether each of the following transformations is a dilation, contraction, rotation, shear, or a combination of these. Also give a rough sketch of the effect of each transformation on $R$.
(i)

$$
\mathrm{V} \rightarrow\left[\begin{array}{cc}
1 / 2 & \sqrt{3} / 2 \\
-\sqrt{3} / 2 & 1 / 2
\end{array}\right] \mathrm{V}
$$

(ii)

$$
\mathbf{V} \rightarrow\left[\begin{array}{ll}
1 & 1 \\
0 & 1
\end{array}\right] \mathbf{V}
$$

(iii)

$$
\mathbf{V} \rightarrow\left[\begin{array}{cc}
2 & 4 \\
-4 & 2
\end{array}\right] \mathbf{V}
$$

(b) Give a linear transformation which squashes $R$ into a rectangle of the same area, but where the width is double the height.
(c) Give a linear transformation which projects $R$ into the line $x=y$.
(d) Show that the result of any linear transformation acting on $R$ is a parallelogram.

