# UNIVERSITY COLLEGE LONDON

*University of London*

## EXAMINATION FOR INTERNAL STUDENTS

*For The Following Qualifications:-*

*B.Sc.* *M.Sci.*

**Mathematics M120: Algebra for Joint Honours Students**

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|---|---|---|
| COURSE CODE | : | **MATHM120** |
| UNIT VALUE | : | **0.50** |
| DATE | : | **02-MAY-03** |
| TIME | : | **14.30** |
| TIME ALLOWED | : | **2 Hours** |

03-C0941-3-50


**TURN OVER**

*All questions may be attempted but only marks obtained on the best* **four** *solutions will count.*
*The use of an electronic calculator is* **not** *permitted in this examination.*

1. Let $G$ be the group with presentation

$$G := \langle r, s \mid r^2 = s^2 = (rs)^4 = e \rangle.$$

Show that $G$ has order 8.

Construct a group table for $G$.

2. Define the term *signum* $\operatorname{sgn} \rho$ of a permutation $\rho \in S_n$, and prove that, if $\tau \in S_n$ is a transposition, then

$$\operatorname{sgn}(\tau\rho) = -\operatorname{sgn} \rho.$$

Let $\rho, \sigma \in S_7$ be

$$\rho := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 3 & 4 & 2 & 6 & 1 & 5 \end{pmatrix}, \qquad \sigma := \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 3 & 1 & 7 & 2 & 5 & 4 \end{pmatrix}.$$

Express $\rho$, $\sigma$, $\rho^{-1}\sigma$ and $\sigma^2\rho^{-3}$ in cycle notation, and find each of their signa.

3. For what values of $\alpha$ does the following system of linear equations have a solution. Find all the solutions when $\alpha$ takes these values.

$$\begin{array}{rcrcrcrcl} 2\xi_1 & - & 4\xi_2 & + & 4\xi_3 & + & 5\xi_4 & = & \alpha \\ -\xi_1 & + & 2\xi_2 & - & 2\xi_3 & + & \alpha\xi_4 & = & 3 \\ 3\xi_1 & - & 6\xi_2 & + & 5\xi_3 & + & 8\xi_4 & = & 2\alpha \\ \xi_1 & - & 2\xi_2 & + & 2\xi_3 & + & 3\xi_4 & = & 1 \end{array}$$

4. Let $A$ be the $4 \times 6$ matrix

$$\begin{bmatrix} 3 & 3 & 9 & 0 & -4 & 2 \\ -1 & -2 & -2 & -2 & 2 & 1 \\ -1 & -4 & 0 & -6 & 9 & 10 \\ -2 & -1 & -7 & 2 & -2 & -7 \end{bmatrix}$$

Find bases for both the row space and the column space of $A$, explaining carefully why your method yields the required answers.

5. Let $M$ be an $n \times n$ matrix over the field $\mathbb{F}$. Show that there exist an invertible matrix $B$ and a diagonal matrix $\Delta$, such that $B^{-1}MB = \Delta$, if and only if $\mathbb{F}^n$ has a basis consisting of eigenvectors of $M$.

Diagonalize, if possible, the matrix

$$M := \begin{bmatrix} 0 & 2 & 2 \\ -2 & -5 & -4 \\ 1 & 2 & 1 \end{bmatrix}.$$

6. What does it mean to say that the set $\{u_1, \ldots, u_k\}$ of vectors in $\mathbb{R}^n$ is *orthonormal*? Prove that an orthonormal set is linearly independent. If $\{u_1, \ldots, u_k\}$ is an orthonormal set and $x$ is a linear combination of $u_1, \ldots, u_k$, explain (with proof) how to find the coefficients in the linear combination in terms of $x$ and $u_1, \ldots, u_k$ alone.

Define what it means to say that an $n \times n$ matrix $U$ is *orthogonal*. Prove that the family $\mathcal{O}_n$ of $n \times n$ orthogonal matrices forms a group.



END OF PAPER