UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C332: Algebra II

COURSE CODE : **MATHC332**

UNIT VALUE : **0.50**

DATE : **18-MAY-06**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

06-C1011-3-40

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Let K be a field, and let α and β be algebraic over K with minimum polynomials of degree a and b respectively. Stating clearly any results you use, prove that $lcm(a,b) \leq [K(\alpha,\beta) : K] \leq ab$. Justifying your answers, give an example of each of the following:

   (i) $lcm(a,b) < [K(\alpha,\beta) : K] = ab$,

   (ii) $lcm(a,b) = [K(\alpha,\beta) : K] < ab$.

   (b) Justifying your answers, find the degree of each of the following extensions:

   (i) $\mathbf{Q}(\alpha) : \mathbf{Q}(\alpha + \frac{3}{\alpha^2})$, where $\alpha = 5^{1/7}$,

   (ii) $\mathbf{Q}(\sqrt{5} + \sqrt{6}) : \mathbf{Q}$.

2. (a) Prove Dedekind's Lemma, i.e. that any set of distinct monomorphisms from one field K to another field L is linearly independent over L.

   (b) What does it mean to say that a group G is *soluble*? If G is soluble and H is a subgroup, prove that H is soluble.

3. (a) Let $L : K$ be a field extension. Define what it means to say that $L : K$ is (i) *normal*, (ii) a *splitting field* for some polynomial f over K. Prove that $L : K$ is a splitting field for some polynomial over K if and only if $L : K$ is a finite normal extension.

   (b) Let $K \subseteq L \subseteq F$ and $K \subseteq M \subseteq F$ for fields K, L, M, F and suppose that $L : K$ and $M : K$ are normal extensions. Prove that $L \cap M : K$ and $LM : K$ are normal extensions. (Here LM denotes the subfield of F generated by L and M.)

4. Suppose that K is an infinite field of characteristic 0, and $L : K$ is a finite extension.

(a) Stating clearly any results that you use, prove that there are only a finite number of fields M such that $K \subseteq M \subseteq L$.

(b) Hence prove that any finite extension of K is simple.

(c) Let $L = \mathbf{Q}(\sqrt[3]{3}, \sqrt[5]{5})$. Find an element $\alpha \in L$ such that $L = \mathbf{Q}(\alpha)$, justifying your answer.

5. Let L be the splitting field of the polynomial $t^7 - 1$ over $\mathbf{Q}$. Find the Galois group G of $L : \mathbf{Q}$. Find all intermediate fields K. Show that one of these intermediate fields is $\mathbf{Q}(\sqrt{-7})$.

[*You should justify your reasoning but may assume relevant results, including the Fundamental Theorem of Galois Theory.*]

END OF PAPER