UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. M.Sci.

Mathematics C332: Algebra II

COURSE CODE : **MATHC332**

UNIT VALUE : **0.50**

DATE : **17-MAY-05**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

05-C0925-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Prove that if $L : K$ and $M : L$ are field extensions of finite degree, then $M : K$ is an extension of finite degree given by $[M : K] = [M : L][L : K]$.

   (b) Justifying your answers, find the following:

   (i) all fields M such that $\mathbf{Q} \subseteq M \subseteq \mathbf{Q}(\sqrt[5]{7})$,

   (ii) $[\mathbf{Q}(\sqrt[3]{2}, \sqrt[4]{3}) : \mathbf{Q}(\sqrt[3]{2})]$.

2. (a) Let G be a finite group of automorphisms of a field K and let K_0 be the fixed field of G. Prove that $[K : K_0] = |G|$. [*You may assume Dedekind's Lemma*]

   (b) $L = \mathbf{C}(t)$ (where t is an indeterminate). Let $\phi : L \longrightarrow L$ be the $\mathbf{C}$-automorphism of L given by $\phi(t) = \frac{t+1}{t-1}$ and let G be the group generated by ϕ. Find the fixed field of G, justifying your answer.

3. Let $L : K$ be a field extension. Define what it means to say that the extension $L : K$ is (a) finite, (b) finitely generated, (c) algebraic, (d) normal (e) a splitting field. State, without proof, the logical relationships that hold between these properties.

   Let $K \subseteq L \subseteq M$ be a tower of fields. For each of the following implications, either prove it, stating any results you use, or provide a counterexample:

   (i) $L : K$ and $M : L$ algebraic $\Rightarrow M : K$ algebraic;

   (ii) $L : K$ and $M : L$ normal $\Rightarrow M : K$ normal

   (iii) $M : K$ normal and finite $\Rightarrow M : L$ normal

   (iv) $M : K$ normal $\Rightarrow M : L$ normal

4. (a) Give the definition of the following:

(i) the extension $L : K$ is *radical*;

(ii) a polynomial $f \in K[t]$ is *soluble by radicals*;

(iii) the *Galois group* of $f(t) \in K[t]$.

State, without proof but defining your terms, the condition on the Galois group of f that is necessary and sufficient for f to be soluble by radicals.

(b) Show that any quartic equation over $\mathbf{Q}$ is soluble by radicals.

(c) Let f be an irreducible quintic over $\mathbf{Q}$ with exactly two non-real roots. Prove that f is not soluble by radicals.

[State clearly any results from group theory that you use in (b) and (c).]

5. Let $\alpha = \sqrt{2 + \sqrt{2}}$ and let $L = \mathbf{Q}(\alpha)$. Show that $L : \mathbf{Q}$ is normal. Find the Galois group and hence find all intermediate fields.

[*You should justify your reasoning but may assume relevant results, including the Fundamental Theorem of Galois Theory.*]