UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C332: Algebra II

COURSE CODE : **MATHC332**

UNIT VALUE : **0.50**

DATE : **17-MAY-04**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

04-C0922-3-30

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Let K be a field, and let α and β be algebraic over K, with minimum polynomials of degree a and b respectively. Stating clearly any results you use, prove that $lcm(a, b) \leq [K(\alpha, \beta) : K] \leq ab$. Give an example where both inequalities are strict, justifying your answer.

   (b) Justifying your answers, find the degree of each of the following extensions:

   (i) $\mathbf{Q}(\alpha) : \mathbf{Q}(\alpha + \frac{2}{\alpha})$, where $\alpha = 2^{1/5}$,

   (ii) $\mathbf{Q}(\alpha, \beta) : \mathbf{Q}$, where $\alpha = 2^{1/5}$ and $\beta = 3^{1/11}$.

2. (a) Prove Dedekind's Lemma (that any finite set of distinct monomorphisms from one field K to another field L is linearly independent over L).

   (b) What does it mean to say that a group G is *soluble*? Prove that if G is soluble and N is a normal subgroup, then G/N is soluble.

3. Let $L : K$ be a field extension. Define what it means to say that the extension $L : K$ is (a) finite, (b) finitely generated, (c) algebraic, (d) normal (e) a splitting field. For each of the following implications, either prove it or provide a counterexample:

   (i) $L : K$ finite $\Rightarrow L : K$ algebraic,

   (ii) $L : K$ finitely generated and algebraic $\Rightarrow L : K$ finite,

   (iii) $L : K$ algebraic $\Rightarrow L : K$ finite,

   (iv) $L : K$ normal and finite $\Rightarrow L : K$ splitting field,

   (v) $L : K$ finite $\Rightarrow L : K$ normal,

4. Suppose that K is a field of characteristic 0, and $L : K$ is a finite extension.

   (a) Stating clearly any results that you use, prove that there are only a finite number of fields M such that $K \subseteq M \subseteq L$.

   (b) Using part (a), prove that any finite extension of K is simple.

   (c) Let $L = \mathbf{Q}(\sqrt{3}, \sqrt[5]{7})$. Find an element $\alpha \in L$ such that $L = \mathbf{Q}(\alpha)$, justifying your answer.

5. Let L be the splitting field of the polynomial $t^5 - 3$ over $\mathbf{Q}$. Find the Galois group G of $L : \mathbf{Q}$ (give generators and relations for G). Find all intermediate fields K of degree 4 over $\mathbf{Q}$.

[*You should justify your reasoning but may assume relevant results, including the Fundamental Theorem of Galois Theory.*]

END OF PAPER