UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C332: Algebra II

COURSE CODE	:	**MATHC332**
UNIT VALUE	:	**0.50**
DATE	:	**06-MAY-03**
TIME	:	**14.30**
TIME ALLOWED	:	**2 Hours**

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TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Prove that if $L : K$ and $M : L$ are field extensions of finite degree, then $M : K$ is an extension of finite degree given by $[M : K] = [M : L][L : K]$.

   (b) Justifying your answers and stating clearly any results you use, find the following:

   (i) $[\mathbf{Q}(\sqrt[5]{5}, \sqrt[7]{7}) : \mathbf{Q}]$,

   (ii) $[\mathbf{Q}(\omega^2 + 1) : \mathbf{Q}]$, where $\omega = e^{2\pi i/5}$.

2. Let G be a finite group of automorphisms of a field K and let K_0 be the fixed field of G. Prove that $[K : K_0] = |G|$. [*You may assume Dedekind's Lemma*]

   If $K = \mathbf{C}(s,t)$ (where s and t are indeterminates) and G is the group of automorphisms of L generated by ϕ, where $\phi(t) = s$, $\phi(s) = t$, find K_0.

3. Let $L : K$ be a field extension. Define what it means to say that (i) $L : K$ is a *normal* extension, (ii) $L : K$ is a *splitting field* for some polynomial f over K. State (do not prove) a condition under which these two properties are equivalent.

   For each of the following extensions, determine whether or not it is normal, justifying your answers.

   (i) $\mathbf{Z}_5(s) : \mathbf{Z}_5(s^5)$, where s is an indeterminate.

   (ii) $\mathbf{Z}_5(s) : \mathbf{Z}_5(s^3)$, where s is an indeterminate.

   (iii) $L_1 \cap L_2 : K$, where $K \subseteq L_i \subseteq F$ $(i = 1, 2)$ for some field F and $L_i : K$ is a normal extension $(i = 1, 2)$.

   (iv) $L_1L_2 : K$, where $K \subseteq L_i \subseteq F$ $(i = 1, 2)$ for some field F and $L_i : K$ is a normal extension $(i = 1, 2)$. (Here L_1L_2 denotes the subfield of F generated by L_1 and L_2.)

4. (a) What does it mean to say that a group G is *soluble*? Prove that if G is soluble, and H is a subgroup of G, then H is also soluble. Show that the group S_5 is not soluble (state any results you assume).

   (b) Let f be an irreducible quintic over $\mathbf{Q}$ with exactly two non-real roots. Stating clearly any results from group theory that you use, prove that the Galois group of f is S_5.

5. Let L be the splitting field of the polynomial $t^6 - 2$ over $\mathbf{Q}$. Find the Galois group G of $L : \mathbf{Q}$. Find all intermediate fields K of degree 4 over $\mathbf{Q}$.

[*You should justify your reasoning but may assume relevant results, including the Fundamental Theorem of Galois Theory.*]