UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For the following qualifications :-

B.Sc. *M.Sci.*

Mathematics C332: Algebra II

COURSE CODE	:	**MATHC332**
UNIT VALUE	:	**0.50**
DATE	:	**13-MAY-02**
TIME	:	**14.30**
TIME ALLOWED	:	**2 hours**

02-C0912-3-40

TURN OVER

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*

The use of an electronic calculator is **not** *permitted in this examination.*

1. (a) Let $L : K$ be a field extension and $\beta \in L$ with minimum polynomial m over K of degree n. Show that every element of $K(\beta)$ can be written uniquely as $g(\beta)$, where g is a polynomial over K of degree strictly less than n. Deduce that $[K(\beta) : K] = n$.

   (b) State (do not prove) the Tower Law. Justifying your answers, find the following:

   (i) $[\mathbf{Q}(\alpha^4 + 3\alpha^2 + 2) : \mathbf{Q}]$, where $\alpha = \sqrt[7]{5}$,

   (ii) $[\mathbf{Q}(\sqrt[3]{1 + \sqrt{p}}) : \mathbf{Q}]$, where p is a prime congruent to 3 (mod 4).

2. (a) Prove Dedekind's Lemma (that any set of distinct monomorphisms from a field K to a field L are linearly independent over L).

   (b) What does it mean to say that a group G is *soluble*? Show that the dihedral group $D_n =< x, y : x^n = y^2 = e, yx = x^{-1}y >$ is soluble. Prove that if G is soluble, and H is a subgroup of G, then H is also soluble.

3. (a) Let $L : K$ be a field extension. Define what it means to say that $L : K$ is (i) *normal*, (ii) a *splitting field* for some polynomial f over K. Prove that if $L : K$ is a splitting field for some polynomial over K then $L : K$ is a normal extension.

   (b) Determine whether each of the following extensions is normal or not, justifying your answers:

   (i) $\mathbf{C}(t) : \mathbf{C}(t^3)$,

   (ii) $\mathbf{C}(t) : \mathbf{C}(t^3 + t + 1)$.

4. Suppose that $L : \mathbf{Q}$ is a finite field extension.

   (a) Stating clearly any results that you use, prove that there are only a finite number of fields M such that $\mathbf{Q} \subseteq M \subseteq L$.

   (b) Hence prove that any finite extension of $\mathbf{Q}$ is simple.

   (c) Let $L = \mathbf{Q}(\sqrt{2}, \sqrt[5]{3})$. Find an element $\alpha \in L$ such that $L = \mathbf{Q}(\alpha)$, justifying your answer.

5. Let L be the splitting field of the polynomial $t^{11} - 1$ over $\mathbf{Q}$. Find the Galois group G of $L : \mathbf{Q}$. Find all intermediate fields K, i.e. fields K such that $\mathbf{Q} \subseteq K \subseteq L$, expressing them as simple extensions of $\mathbf{Q}$.

[*You should justify your reasoning but may assume relevant results, including the Fundamental Theorem of Galois Theory.*]

END OF PAPER