UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C331: Algebra I

COURSE CODE : **MATHC331**

UNIT VALUE : **0.50**

DATE : **04-MAY-06**

TIME : **10.00**

TIME ALLOWED : **2 Hours**

06-C1010-3-40

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination.* Throughout all rings are assumed to be commutative with a 1 and all free modules are assumed to be of finite rank.

1. Define the following terms for an integral domain R.

   (i) Euclidean domain, (ii) principal ideal domain, (iii) unique factorization domain. State, **without proof**, the relationship between these properties.

   (a) If (R, N) is a Euclidean domain, show that R is a principal ideal domain.

   (b) Let $\mathbb{Z}[i]$ be the integral domain $\{a + bi : a, b \in \mathbb{Z}\} \subseteq \mathbb{C}$. Define $N : \mathbb{Z}[i] \to \mathbb{N}$ by $a + bi \mapsto a^2 + b^2$. Show that $(\mathbb{Z}[i], N)$ is a Euclidean domain.

   (c) Find a factorization of $25 + 5i \in \mathbb{Z}[i]$ into atoms. Justify your answer.

   [For your answers to (b) and (c), you may assume any of the usual properties of $\mathbb{Z}[i]$.]

2. Define what it means to say that $\alpha : M \to N$ is an R-module homomorphism for R-modules M and N. If this is the case, show that for such α

   (i) $\text{Im}(\alpha) \leqslant N$, (ii) $\text{Ker}(a) \leqslant M$, (iii) $\text{Im}(\alpha) \cong M/\text{Ker}(\alpha)$.

   (a) Define what it means to say that an R-module M is cyclic. Show that M is cyclic if and only if $M \cong R/I$ for some ideal I of R. Show that I is unique.

   (b) A non-zero R-module M is said to be **irreducible** if O and M are its only submodules. Show that if M is irreducible, then M is cyclic, and so show that M is irreducible if and only if $M \cong R/I$ for some maximal ideal of R.

   [You may use any standard results about modules in your answers to (a) and (b).]

3. (a) Define the terms (i) free module, (ii) basis for a module, and explain, **without proof**, the relationship between them.

Let F be a free module and M an R-module. If $\{e_1, ..., e_n\}$ is a basis for F and $\{m_1, ..., m_n\} \subseteq M$, show that there is a unique R-homomorphism $\alpha : F \to M$ such that $\alpha(e_i) = m_i$, $1 \leqslant i \leqslant n$. If M is generated by $\{m_1, ..., m_n\}$, show that $M \cong F/P$ for some submodule P of M.

(b) Let R be a principal ideal domain and $A \in {}^mR^n$. Define the Smith Normal Form of A over R and state the extent to which it is unique.

Let $A = \begin{bmatrix} 6 & 4 & 4 \\ 4 & 4 & 8 \\ 8 & 6 & 10 \end{bmatrix} \in {}^3\mathbb{Z}^3$. Find $P, Q \in GL(3, \mathbb{Z})$ such that PAQ is in Smith Normal Form over $\mathbb{Z}$.

4. State, **without proof**, the classification of finitely generated abelian groups indicating to what extent this classification is unique.

(a) Let $p \in \mathbb{N}$ be a prime. Explain how to find all abelian groups of order p^n. How many distinct abelian groups are there of order (i) 5^6,(ii) 6^5. Justify your answers.

[You are not expected to find the actual groups themselves.]

(b) Classify all abelian groups of order 1080 by means of both invariant factors and elementary divisors.

(c) Find the abelian group given by the generators e_1, e_2, e_3, e_4, e_5 subject to the relations:

$$\begin{array}{ccccccccccc} e_1 & + & e_2 & + & e_3 & + & 2e_4 & & & = & 0 \\ e_1 & + & 3e_2 & + & e_3 & + & 4e_4 & + & 2e_5 & = & 0 \\ & - & 2e_2 & & & - & 2e_4 & - & 2e_5 & = & 0 \\ e_1 & + & 3e_2 & + & 7e_3 & + & 4e_4 & + & 2e_5 & = & 0 \end{array}.$$

5. (a) Let $\alpha : V \to V$ be a linear map, where V is a finite-dimensional vector space over a field $\mathbb{F}$. Explain how V can be considered as a finitely generated torsion module over $\mathbb{F}[x]$.

(b) Let $A \in {}^n\mathbb{C}^n$. Write a **brief** essay **without proofs** explaining how the structure theory of finitely generated modules over principal ideal domains can be used to classify A under similarity. Your essay should contain, in particular, the definitions of the Rational Canonical Form (RCF) and Jordan Normal Form (JNF) of A and explain their relationships to this classification.

(c) Let $A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in {}^4\mathbb{C}^4$. By reducing the characteristic matrix of A appropriately, find the RCF and JNF of A.

END OF PAPER