UNIVERSITY COLLEGE LONDON

University of London

EXAMINATION FOR INTERNAL STUDENTS

For The Following Qualifications:-

B.Sc. *M.Sci.*

Mathematics C331: Algebra I

COURSE CODE : **MATHC331**

UNIT VALUE : **0.50**

DATE : **09-MAY-05**

TIME : **14.30**

TIME ALLOWED : **2 Hours**

05-C0924-3-40

TURN OVER

All *questions may be attempted but only marks obtained on the best* **four** *solutions will count. The use of an electronic calculator is* **not** *permitted in this examination. Throughout all rings are commutative with a 1 and and all free modules are assumed to be of finite rank.*

1. Define the following terms for the integral domain R :

   (i) atom, (ii) prime, (iii) unique factorization domain.

   (a) Show that every prime is an atom.

   (b) If every non-zero, non-unit element of R is a product of primes, show that R is a unique factorization domain.

   (c) Find all the factorizations of 16 in $\mathbb{Z}[\sqrt{-7}]$ as a product of atoms to within associates and to within order. Hence show that $\mathbb{Z}[\sqrt{-7}]$ is not a unique factorization domain and that, in particular, 16 is not the product of primes.
   [You may assume, without proof, any standard results about the integral domains $\mathbb{Z}[\sqrt{m}]$ where $m \in \mathbb{Z}$ and $m \leqslant -1$.]

2. Let ${}_RM$ be an R-module. State what it means to say that P is a submodule of M. Describe, *without proof*, how M/P can be given an R-module structure and state, *without proof*, what are the submodules of M/P.

   (a) What does it mean to say that M is finitely generated? Show that if this is the case, then M/P is also finitely generated.

   (b) Suppose that both P and M/P are finitely generated. Show that M is finitely generated.

   (c) Show that every submodule of a free module ${}_RM$ over a principal ideal domain R is finitely generated with a generating set with at most n elements, where $r(M) = n$. Hence or otherwise show that every submodule of a finitely generated module over a principal ideal domain is finitely generated.
   [For (c) you may assume any standard results about free modules, which you require for your proof.]

3. Let (R, N) be a Euclidean domain and $A \in {}^mR^n$. Define the Smith Normal Form of A and state to what extent it is unique. Describe, *with proof,* how to reduce A to Smith Normal Form. Briefly describe, *without proof,* how to show that the Smith Normal Form of A is unique.

Let $A = \begin{bmatrix} 0 & 2 & 4 \\ 2 & 4 & 6 \\ 2 & 6 & 10 \\ -2 & 0 & 7 \end{bmatrix} \in {}^4\mathbb{Z}^3$. Find the Smith Normal Form of A.

4. (a) Define the torsion submodule $T(M)$ of a module M over an integral domain R. Show that $T(M)$ is actually a submodule of M.

(b) State, without proof, the classification of finitely generated modules ${}_RM$ over a principal ideal domain R. Identify, *with proof,* the torsion modules in this classification.

(c) Let M be a finitely generated abelian group. Show that M is a torsion group if and only if M is finite.

(d) Find all non-isomorphic abelian groups of order 3000 by means of both elementary divisors and invariant factors.

In this classification identify the following abelian groups of order 3000 stating which are isomorphic to each other.

(i) $\mathbb{Z}_5 \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_{100}$, (ii) $\mathbb{Z}_{40} \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_5$, (iii) $\mathbb{Z}_{10} \oplus \mathbb{Z}_{15} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{10}$,
(iv) $\mathbb{Z}_{500} \oplus \mathbb{Z}_6$, (v) $\mathbb{Z}_3 \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{10} \oplus \mathbb{Z}_{10}$, (vi) $\mathbb{Z}_{125} \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_4$.

5. (a) Define the companion matrix C_d of $d \in \mathbb{F}[x]$ where d is a monic polynomial of degree $\geqslant 1$ and $\mathbb{F}$ is a field.

(b) Let $\alpha : V \to V$ be a linear map, where V is a finite-dimensional vector space over $\mathbb{F}$. Explain, without proof, how V can be considered as a finitely generated torsion module over $\mathbb{F}[x]$.

(c) Suppose that $\alpha : V \to V$ gives rise to a cyclic $\mathbb{F}[x]$ module $V = \mathbb{F}[x]\underline{v} \cong \mathbb{F}[x]/(d)$, where $\underline{v} \in V$ and d is the minimum polynomial of α. Show that V has a basis e such that $[\alpha]^e_e = C_d$.

(d) Let $A = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in {}^4\mathbb{C}^4$. By reducing the characteristic matrix of A to Smith Normal Form, find the Rational Canonical Form and the Jordan Normal Form of A.

END OF PAPER